Problem 16

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} = \frac{e^{-x}\cos y - e^{2y}\cos x}{-e^{-x}\sin y + 2e^{2y}\sin x}$$

Solution

Write the ODE as $M(x, y) + N(x, y)y' = 0$.

$$(-e^{-x}\sin y + 2e^{2y}\sin x)\frac{dy}{dx} = e^{-x}\cos y - e^{2y}\cos x$$

$$(e^{2y}\cos x - e^{-x}\cos y) + (-e^{-x}\sin y + 2e^{2y}\sin x)\frac{dy}{dx} = 0 \tag{1}$$

This ODE is exact because

$$\frac{\partial}{\partial y}(e^{2y}\cos x - e^{-x}\cos y) = \frac{\partial}{\partial x}(-e^{-x}\sin y + 2e^{2y}\sin x) = e^{-x}\sin y + 2e^{2y}\cos x.$$

That means there exists a potential function $\psi = \psi(x, y)$ which satisfies

$$\frac{\partial\psi}{\partial x} = e^{2y}\cos x - e^{-x}\cos y \tag{2}$$

$$\frac{\partial\psi}{\partial y} = -e^{-x}\sin y + 2e^{2y}\sin x. \tag{3}$$

Integrate both sides of equation (2) partially with respect to x to get ψ.

$$\psi(x, y) = e^{2y}\sin x + e^{-x}\cos y + f(y)$$

Here $f(y)$ is an arbitrary function of y. Differentiate both sides with respect to y.

$$\psi_y(x, y) = 2e^{2y}\sin x - e^{-x}\sin y + f'(y)$$

Comparing this to equation (3), we see that

$$f'(y) = 0 \quad \rightarrow \quad f(y) = 0.$$

Consequently, a potential function is

$$\psi(x, y) = e^{2y}\sin x + e^{-x}\cos y.$$

Notice that by substituting equations (2) and (3), equation (1) can be written as

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial\psi}{\partial x}\,dx + \frac{\partial\psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}$$

With it, equation (4) becomes

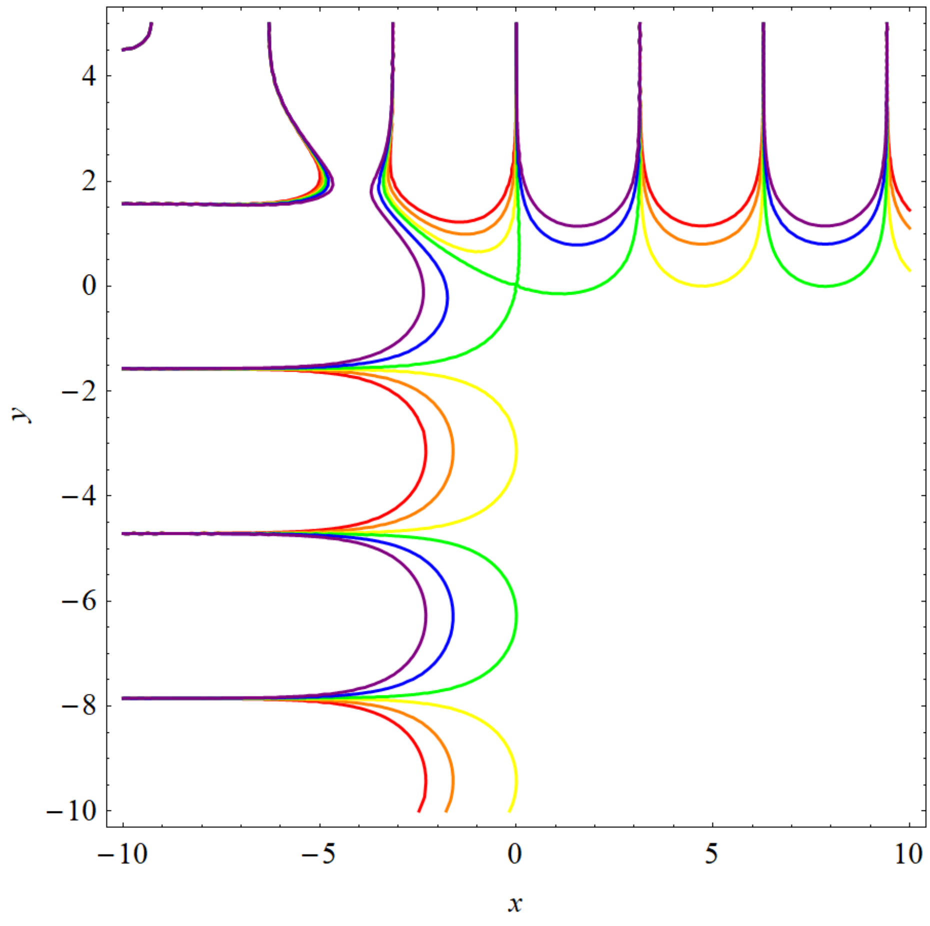
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

Therefore,

$$e^{2y}\sin x + e^{-x}\cos y = C.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.