Problem 17

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} = e^{2x} + 3y$$

Solution

Method Using an Integrating Factor I

Bring $3y$ to the left side.

$$\frac{dy}{dx} - 3y = e^{2x}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^x (-3)\, ds\right) = e^{-3x}$$

Proceed with the multiplication.

$$e^{-3x}\frac{dy}{dx} - 3e^{-3x}y = e^{2x}e^{-3x}$$

The left side can be written as $d/dx(Iy)$ by the chain rule.

$$\frac{d}{dx}(e^{-3x}y) = e^{-x}$$

Integrate both sides with respect to x.

$$e^{-3x}y = -e^{-x} + C$$

Therefore,

$$y(x) = -e^{2x} + Ce^{3x}.$$

Method Using an Integrating Factor II

$$\frac{dy}{dx} = e^{2x} + 3y$$

Write the ODE as $M(x, y) + N(x, y)y' = 0$.

$$(-e^{2x} - 3y) + \frac{dy}{dx} = 0 \tag{1}$$

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(-e^{2x} - 3y) = -3 \neq \frac{\partial}{\partial x}(1) = 0.$$

To solve it, we seek an integrating factor $\mu = \mu(x, y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$(-e^{2x} - 3y)\mu + \mu\frac{dy}{dx} = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}[(-e^{2x} - 3y)\mu] = \frac{\partial}{\partial x}(\mu).$$

Expand both sides.

$$-3\mu + (-e^{2x} - 3y)\frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x}$$

Assume that μ is only dependent on x: $\mu = \mu(x)$.

$$-3\mu = \frac{d\mu}{dx}$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = -3\,dx$$

Integrate both sides.

$$\ln \mu = -3x + C_1$$

Exponentiate both sides.

$$\begin{aligned}\mu &= e^{-3x+C_1} \\ &= e^{-3x}e^{C_1}\end{aligned}$$

Taking e^{C_1} to be 1, an integrating factor is

$$\mu = e^{-3x}.$$

Multiply both sides of equation (1) by e^{-3x}.

$$(-e^{-x} - 3e^{-3x}y) + e^{-3x}\frac{dy}{dx} = 0 \tag{2}$$

Because it's exact now, there exists a potential function $\psi = \psi(x, y)$ which satisfies

$$\frac{\partial\psi}{\partial x} = -e^{-x} - 3e^{-3x}y \tag{3}$$

$$\frac{\partial\psi}{\partial y} = e^{-3x}. \tag{4}$$

Integrate both sides of equation (4) partially with respect to y to get ψ.

$$\psi(x, y) = e^{-3x}y + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x, y) = -3e^{-3x}y + f'(x)$$

Comparing this to equation (3), we see that

$$f'(x) = -e^{-x} \quad \rightarrow \quad f(x) = e^{-x}.$$

Consequently, a potential function is

$$\psi(x, y) = e^{-3x}y + e^{-x}.$$

Notice that by substituting equations (3) and (4), equation (2) can be written as

$$\frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 0. \tag{5}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial\psi}{\partial x}\,dx + \frac{\partial\psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx}$$

With it, equation (5) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

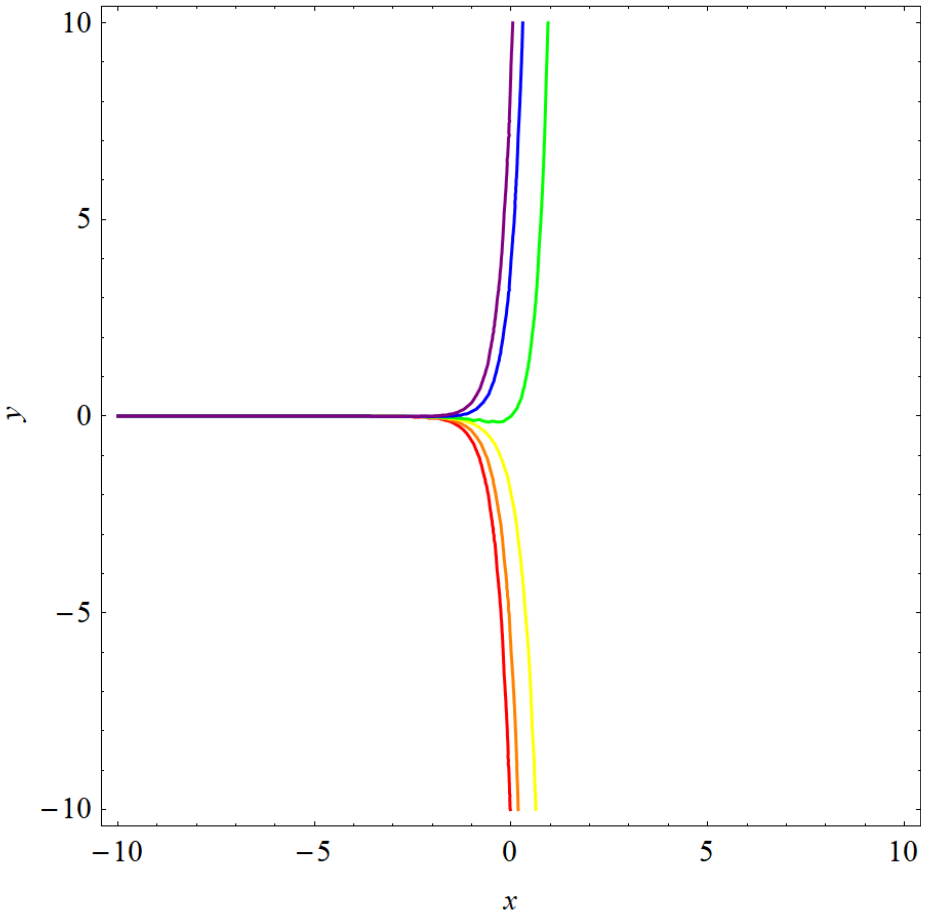
$$\psi(x, y) = C_2$$

Therefore,

$$e^{-3x}y + e^{-x} = C_2,$$

or solving for y explicitly,

$$y(x) = C_2e^{3x} - e^{2x}.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.