Problem 17

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

\[ \frac{dy}{dx} = e^{2x} + 3y \]

Solution

Method Using an Integrating Factor I

Bring 3y to the left side.

\[ \frac{dy}{dx} - 3y = e^{2x} \]

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor \( I \).

\[ I = \exp \left( \int (-3) \, ds \right) = e^{-3x} \]

Proceed with the multiplication.

\[ e^{-3x} \frac{dy}{dx} - 3e^{-3x}y = e^{2x}e^{-3x} \]

The left side can be written as \( d/dx(ly) \) by the chain rule.

\[ \frac{d}{dx}(e^{-3x}y) = e^{-x} \]

Integrate both sides with respect to \( x \).

\[ e^{-3x}y = -e^{-x} + C \]

Therefore,

\[ y(x) = -e^{2x} + Ce^{3x}. \]
Method Using an Integrating Factor II

\[ \frac{dy}{dx} = e^{2x} + 3y \]

Write the ODE as \( M(x, y) + N(x, y)y' = 0 \).

\[ (-e^{2x} - 3y) + \frac{dy}{dx} = 0 \quad (1) \]

This ODE is not exact at the moment because

\[ \frac{\partial}{\partial y}(-e^{2x} - 3y) = -3 \neq \frac{\partial}{\partial x}(1) = 0. \]

To solve it, we seek an integrating factor \( \mu = \mu(x, y) \) such that when both sides are multiplied by it, the ODE becomes exact.

\[ (-e^{2x} - 3y)\mu + \mu \frac{dy}{dx} = 0 \]

Since the ODE is exact now,

\[ \frac{\partial}{\partial y} [(-e^{2x} - 3y)\mu] = \frac{\partial}{\partial x}(\mu). \]

Expand both sides.

\[ -3\mu + (-e^{2x} - 3y) \frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x} \]

Assume that \( \mu \) is only dependent on \( x \): \( \mu = \mu(x) \).

\[ -3\mu = \frac{d\mu}{dx} \]

Solve this ODE by separating variables.

\[ \frac{d\mu}{\mu} = -3 \, dx \]

Integrate both sides.

\[ \ln \mu = -3x + C_1 \]

Exponentiate both sides.

\[ \mu = e^{-3x+C_1} \]

\[ = e^{-3x}e^{C_1} \]

Taking \( e^{C_1} \) to be 1, an integrating factor is

\[ \mu = e^{-3x}. \]

Multiply both sides of equation (1) by \( e^{-3x} \).

\[ (-e^{-x} - 3e^{-3x}y) + e^{-3x} \frac{dy}{dx} = 0 \quad (2) \]

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Because it’s exact now, there exists a potential function \( \psi = \psi(x, y) \) which satisfies

\[
\frac{\partial \psi}{\partial x} = -e^{-x} - 3e^{-3x}y \quad (3)
\]
\[
\frac{\partial \psi}{\partial y} = e^{-3x}. \quad (4)
\]

Integrate both sides of equation (4) partially with respect to \( y \) to get \( \psi \).

\[
\psi(x, y) = e^{-3x}y + f(x)
\]

Here \( f(x) \) is an arbitrary function of \( x \). Differentiate both sides with respect to \( x \).

\[
\psi_x(x, y) = -3e^{-3x}y + f'(x)
\]

Comparing this to equation (3), we see that

\[
f'(x) = -e^{-x} \quad \rightarrow \quad f(x) = e^{-x}.
\]

Consequently, a potential function is

\[
\psi(x, y) = e^{-3x}y + e^{-x}.
\]

Notice that by substituting equations (3) and (4), equation (2) can be written as

\[
\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \quad (5)
\]

Recall that the differential of \( \psi(x, y) \) is defined as

\[
d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.
\]

Dividing both sides by \( dx \), we obtain the fundamental relationship between the total derivative of \( \psi \) and its partial derivatives.

\[
\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}
\]

With it, equation (5) becomes

\[
\frac{d\psi}{dx} = 0.
\]

Integrate both sides with respect to \( x \).

\[
\psi(x, y) = C_2
\]

Therefore,

\[
e^{-3x}y + e^{-x} = C_2,
\]

or solving for \( y \) explicitly,

\[
y(x) = C_2e^{3x} - e^{2x}.
\]
This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.