Problem 18

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} + 2y = e^{-x^2-2x}, \qquad y(0) = 3$$

Solution

Method Using an Integrating Factor I

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^x 2\,ds\right) = e^{2x}$$

Proceed with the multiplication.

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y = e^{-x^2-2x}e^{2x}$$

The left side can be written as $d/dx(Iy)$ by the chain rule.

$$\frac{d}{dx}(e^{2x}y) = e^{-x^2}$$

Integrate both sides with respect to x.

$$e^{2x}y = \int^x e^{-s^2}\,ds + C$$

Because C is arbitrary, the lower limit of integration is as well. C will be adjusted to account for whatever choice we make when we apply the boundary condition.

$$e^{2x}y = \int_0^x e^{-s^2}\,ds + C$$

Divide both sides by e^{2x}.

$$y(x) = e^{-2x}\left(\int_0^x e^{-s^2}\,ds + C\right)$$

Apply the boundary condition $y(0) = 3$ to determine C.

$$y(0) = 1\left(\int_0^0 e^{-s^2}\,ds + C\right) = 3 \quad \to \quad C = 3$$

Therefore,

$$y(x) = e^{-2x}\left(\int_0^x e^{-s^2}\,ds + 3\right).$$

Method Using an Integrating Factor II

$$\frac{dy}{dx} + 2y = e^{-x^2-2x}$$

Write the ODE as $M(x,y) + N(x,y)y' = 0$.

$$(2y - e^{-x^2-2x}) + \frac{dy}{dx} = 0 \tag{1}$$

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(2y - e^{-x^2-2x}) = 2 \neq \frac{\partial}{\partial x}(1) = 0.$$

To solve it, we seek an integrating factor $\mu = \mu(x,y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$(2y - e^{-x^2-2x})\mu + \mu\frac{dy}{dx} = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}[(2y - e^{-x^2-2x})\mu] = \frac{\partial}{\partial x}(\mu).$$

Expand both sides.

$$2\mu + (2y - e^{-x^2-2x})\frac{\partial \mu}{\partial y} = \frac{\partial \mu}{\partial x}$$

Assume that μ is only dependent on x: $\mu = \mu(x)$.

$$2\mu = \frac{d\mu}{dx}$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu} = 2\,dx$$

Integrate both sides.

$$\ln \mu = 2x + C_1$$

Exponentiate both sides.

$$\begin{aligned}\mu &= e^{2x+C_1} \\ &= e^{2x}e^{C_1}\end{aligned}$$

Taking e^{C_1} to be 1, an integrating factor is

$$\mu = e^{2x}.$$

Multiply both sides of equation (1) by e^{2x}.

$$(2e^{2x}y - e^{-x^2}) + e^{2x}\frac{dy}{dx} = 0 \tag{2}$$

Because it's exact now, there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = 2e^{2x}y - e^{-x^2} \tag{3}$$

$$\frac{\partial \psi}{\partial y} = e^{2x}. \tag{4}$$

Integrate both sides of equation (4) partially with respect to y to get ψ.

$$\psi(x, y) = e^{2x}y + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x, y) = 2e^{2x}y + f'(x)$$

Comparing this to equation (3), we see that

$$f'(x) = -e^{-x^2} \quad \rightarrow \quad f(x) = -\int_0^x e^{-s^2}\,ds.$$

Consequently, a potential function is

$$\psi(x, y) = e^{2x}y - \int_0^x e^{-s^2}\,ds.$$

Notice that by substituting equations (3) and (4), equation (2) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{5}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (5) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C_2$$

The general solution is then

$$e^{2x}y - \int_0^x e^{-s^2}\,ds = C_2,$$

or solving for y explicitly,

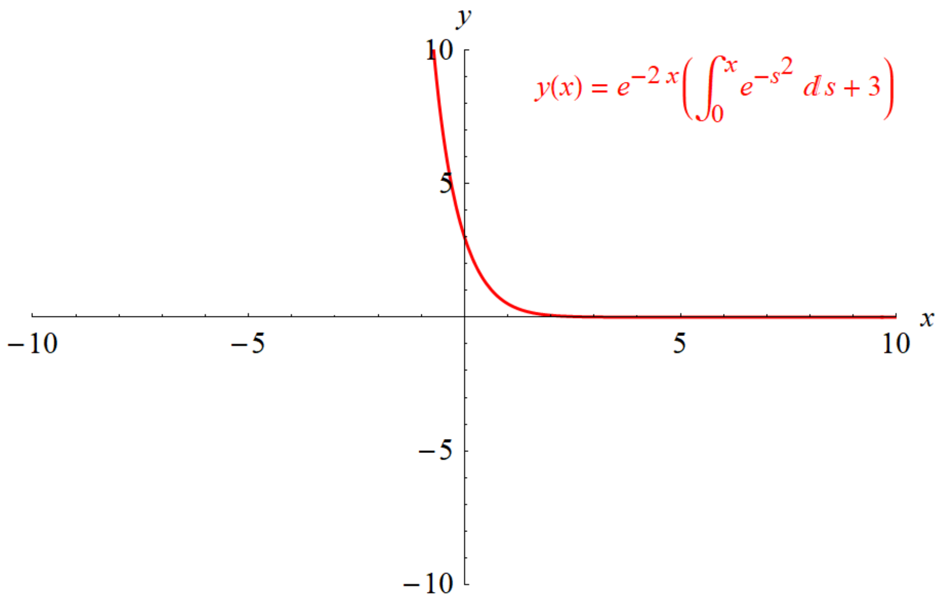
$$y(x) = e^{-2x}\left(\int_0^x e^{-s^2}\,ds + C_2\right).$$

Apply the boundary condition $y(0) = 3$ to determine C_2.

$$y(0) = 1\left(\int_0^0 e^{-s^2}\,ds + C_2\right) = 3 \quad \rightarrow \quad C_2 = 3$$

Therefore,

$$y(x) = e^{-2x}\left(\int_0^x e^{-s^2}\,ds + 3\right).$$



This figure illustrates the solution to the ODE in the xy-plane that passes through the point $(0, 3)$.