Problem 22

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

\[ \frac{dy}{dx} = \frac{x^2 - 1}{y^2 + 1}, \quad y(-1) = 1 \]

Solution

Method Using Separation of Variables

Because the ODE is of the form \( y' = f(x)/g(y) \), it can be solved by separating variables.

\[ (y^2 + 1) \, dy = (x^2 - 1) \, dx \]

Integrate both sides.

\[ \int (y^2 + 1) \, dy = \int (x^2 - 1) \, dx \]

\[ \frac{y^3}{3} + y = \frac{x^3}{3} - x + C \]

The general solution is then

\[ \frac{y^3}{3} + y - \frac{x^3}{3} + x = C. \]

Apply the boundary condition \( y(-1) = 1 \) now to determine \( C \).

\[ \frac{1}{3} + 1 - \frac{(-1)^3}{3} + (-1) = C \quad \Rightarrow \quad C = \frac{2}{3} \]

Therefore,

\[ \frac{y^3}{3} + y - \frac{x^3}{3} + x = \frac{2}{3}. \]
Method Using an Integrating Factor

\[ \frac{dy}{dx} = \frac{x^2 - 1}{y^2 + 1} \]

Write the ODE as \( M(x, y) + N(x, y)y' = 0 \).

\[(y^2 + 1) \frac{dy}{dx} = x^2 - 1 \]

\[(1 - x^2) + (y^2 + 1) \frac{dy}{dx} = 0 \] (1)

This ODE is exact because

\[ \frac{\partial}{\partial y}(1 - x^2) = \frac{\partial}{\partial x}(y^2 + 1) = 0. \]

That means there exists a potential function \( \psi = \psi(x, y) \) which satisfies

\[ \frac{\partial \psi}{\partial x} = 1 - x^2 \] (2)

\[ \frac{\partial \psi}{\partial y} = y^2 + 1. \] (3)

Integrate both sides of equation (2) partially with respect to \( x \) to get \( \psi \).

\[ \psi(x, y) = x - \frac{x^3}{3} + f(y) \]

Here \( f(y) \) is an arbitrary function of \( y \). Differentiate both sides with respect to \( y \).

\[ \psi_y(x, y) = f'(y) \]

Comparing this to equation (3), we see that

\[ f'(y) = y^2 + 1 \rightarrow f(y) = \frac{y^3}{3} + y. \]

As a result, a potential function is

\[ \psi(x, y) = x - \frac{x^3}{3} + \frac{y^3}{3} + y. \]

Notice that by substituting equations (2) and (3), equation (1) can be written as

\[ \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \] (4)

Recall that the differential of \( \psi(x, y) \) is defined as

\[ d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy. \]

Dividing both sides by \( dx \), we obtain the fundamental relationship between the total derivative of \( \psi \) and its partial derivatives.

\[ \frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} \]

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With it, equation (4) becomes
\[ \frac{d\psi}{dx} = 0. \]
Integrate both sides with respect to \( x \).
\[ \psi(x, y) = C_1 \]
The general solution is then
\[ x - \frac{x^3}{3} + \frac{y^3}{3} + y = C_1. \]
Apply the boundary condition \( y(-1) = 1 \) now to determine \( C_1 \).
\[ (-1) - \frac{(-1)^3}{3} + \frac{1}{3} + 1 = C_1 \quad \rightarrow \quad C_1 = \frac{2}{3} \]
Therefore,
\[ x - \frac{x^3}{3} + \frac{y^3}{3} + y = \frac{2}{3}. \]

This figure illustrates the solution to the ODE in the \( xy \)-plane that passes through the point \((-1, 1)\).