Problem 23

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$t\frac{dy}{dt} + (t+1)y = e^{2t}$$

Solution

Method Using an Integrating Factor I

Divide both sides by t.

$$\frac{dy}{dt} + \left(1 + \frac{1}{t}\right) y = \frac{e^{2t}}{t}$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left[\int^t \left(1 + \frac{1}{s}\right) ds\right] = e^{t+\ln t} = e^t e^{\ln t} = te^t$$

Proceed with the multiplication.

$$te^t\frac{dy}{dt} + \left(1 + \frac{1}{t}\right) te^t y = \frac{e^{2t}}{t} te^t$$

$$te^t\frac{dy}{dt} + (t+1)e^t y = e^{3t}$$

The left side can be written as $d/dt(Iy)$ by the chain rule.

$$\frac{d}{dt}(te^t y) = e^{3t}$$

Integrate both sides with respect to t.

$$te^t y = \frac{e^{3t}}{3} + C$$

Therefore, dividing both sides by te^t,

$$y(t) = \frac{e^{2t}}{3t} + \frac{C}{te^t}.$$

Method Using an Integrating Factor II

$$t\frac{dy}{dt} + (t+1)y = e^{2t}$$

Write the ODE as $M(y,t) + N(y,t)y' = 0$.

$$(ty + y - e^{2t}) + t\frac{dy}{dt} = 0 \tag{1}$$

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(ty + y - e^{2t}) = t + 1 \neq \frac{\partial}{\partial t}(t) = 1.$$

To solve it, we seek an integrating factor $\mu = \mu(y,t)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$(ty + y - e^{2t})\mu + t\mu\frac{dy}{dt} = 0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}[(ty + y - e^{2t})\mu] = \frac{\partial}{\partial t}(t\mu).$$

Expand both sides.

$$(t+1)\mu + (ty + y - e^{2t})\frac{\partial \mu}{\partial y} = \mu + t\frac{\partial \mu}{\partial t}$$

Assume that μ is only dependent on t: $\mu = \mu(t)$.

$$(t+1)\mu = \mu + t\frac{d\mu}{dt}$$

$$t\mu = t\frac{d\mu}{dt}$$

$$\frac{d\mu}{dt} = \mu$$

An integrating factor is

$$\mu = e^t.$$

Multiply both sides of equation (1) by e^t.

$$(te^t y + e^t y - e^{3t}) + te^t\frac{dy}{dt} = 0 \tag{2}$$

Because it's exact now, there exists a potential function $\psi = \psi(y,t)$ that satisfies

$$\frac{\partial \psi}{\partial t} = te^t y + e^t y - e^{3t} \tag{3}$$

$$\frac{\partial \psi}{\partial y} = te^t. \tag{4}$$

Integrate both sides of equation (4) partially with respect to y to get ψ.

$$\psi(y,t) = te^t y + f(t)$$

Here $f(t)$ is an arbitrary function of t. Differentiate both sides with respect to t.

$$\psi_t(y,t) = (t+1)e^t y + f'(t)$$

Comparing this to equation (3), we see that

$$f'(t) = -e^{3t} \quad \rightarrow \quad f(t) = -\frac{e^{3t}}{3}.$$

Consequently, a potential function is

$$\psi(y,t) = te^t y - \frac{e^{3t}}{3}$$

Notice that by substituting equations (3) and (4), equation (2) can be written as

$$\frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial y}\frac{dy}{dt} = 0. \tag{5}$$

Recall that the differential of $\psi(y,t)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial t}\,dt + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dt, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dt} = \frac{\partial \psi}{\partial t} + \frac{\partial \psi}{\partial y}\frac{dy}{dt}$$

With it, equation (5) becomes

$$\frac{d\psi}{dt} = 0.$$

Integrate both sides with respect to t.

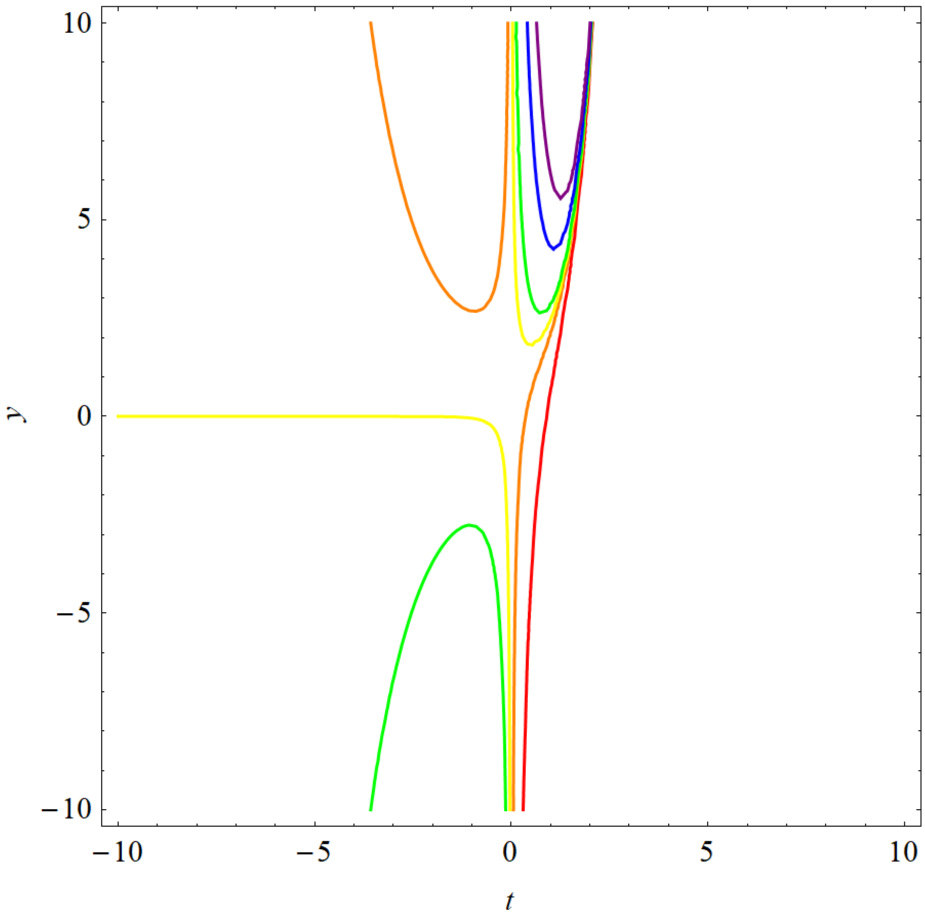
$$\psi(y,t) = C_1$$

Therefore,

$$te^t y - \frac{e^{3t}}{3} = C_1,$$

or solving for y explicitly,

$$y(t) = \frac{e^{2t}}{3t} + \frac{C_1}{te^t}.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -5$, $C = -1$, $C = 0$, $C = 1$, $C = 5$, and $C = 10$, respectively.