# Problem 24

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$2\sin y\sin x\cos x+\cos y\sin^2 x\frac{dy}{dx}=0$$

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**Solution**

**Method Using Separation of Variables**

Divide both sides by $\sin x$

$$2\sin y\cos x+\cos y\sin x\frac{dy}{dx}=0$$

and solve for $dy/dx$.

$$\frac{dy}{dx}=-\frac{2\sin y\cos x}{\cos y\sin x}=-\frac{\sin y}{\cos y}\cdot\frac{2\cos x}{\sin x}$$

Because the ODE is of the form $y'=f(x)g(y)$, it can be solved by separating variables.

$$\frac{\cos y}{\sin y}\,dy=-2\frac{\cos x}{\sin x}\,dx$$

Integrate both sides.

$$\int\frac{\cos y}{\sin y}\,dy=\int -2\frac{\cos x}{\sin x}\,dx$$

$$\ln|\sin y|=-2\ln|\sin x|+C$$

Isolate $C$ on the right side.

$$\ln|\sin y|+2\ln|\sin x|=C$$

$$\ln|\sin y|+\ln\sin^2 x=C$$

$$\ln|\sin y|\sin^2 x=C$$

Exponentiate both sides.

$$|\sin y|\sin^2 x=e^C$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$\sin y\sin^2 x=\pm e^C$$

Therefore, using a new constant $A$ for the right side,

$$\sin y\sin^2 x=A.$$

**Method Using an Integrating Factor**

$$2\sin y \sin x \cos x + \cos y \sin^2 x \frac{dy}{dx} = 0 \tag{1}$$

This ODE is exact because

$$\frac{\partial}{\partial y}(2\sin y \sin x \cos x) = \frac{\partial}{\partial x}(\cos y \sin^2 x) = 2\cos y \sin x \cos x.$$

That means there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = 2\sin y \sin x \cos x \tag{2}$$

$$\frac{\partial \psi}{\partial y} = \cos y \sin^2 x. \tag{3}$$

Integrate both sides of equation (3) partially with respect to $y$ to get $\psi$.

$$\psi(x, y) = \sin y \sin^2 x + f(x)$$

Here $f(x)$ is an arbitrary function of $x$. Differentiate both sides with respect to $x$.

$$\psi_y(x, y) = 2\sin y \sin x \cos x + f'(x)$$

Comparing this to equation (2), we see that

$$f'(x) = 0 \quad \rightarrow \quad f(x) = 0.$$

As a result, a potential function is

$$\psi(x, y) = \sin y \sin^2 x.$$

Notice that by substituting equations (2) and (3), equation (1) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{4}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by $dx$, we obtain the fundamental relationship between the total derivative of $\psi$ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (4) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to $x$.

$$\psi(x, y) = C_1$$

Therefore,

$$\sin y \sin^2 x = C_1.$$

## The Easy Way

$$2\sin y \sin x \cos x + \cos y \sin^2 x \frac{dy}{dx} = 0$$

Notice that the ODE can be written as

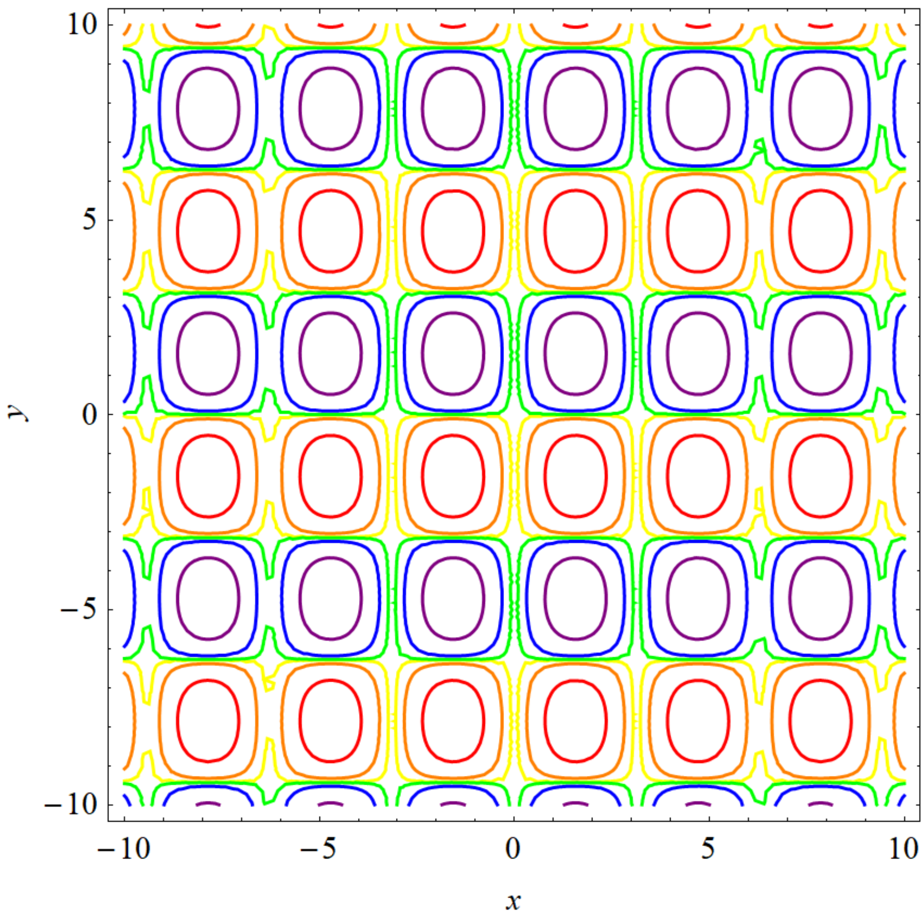
$$\sin y \frac{d}{dx}(\sin^2 x) + \frac{d}{dy}(\sin y)\sin^2 x \frac{dy}{dx} = 0$$

$$\frac{\partial}{\partial x}(\sin y \sin^2 x) + \frac{\partial}{\partial y}(\sin y \sin^2 x)\frac{dy}{dx} = 0$$

$$\frac{d}{dx}(\sin y \sin^2 x) = 0.$$

Therefore, integrating both sides with respect to $x$,

$$\sin y \sin^2 x = B.$$




This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $B = -0.5$, $B = -0.1$, $B = -0.01$, $B = 0.01$, $B = 0.1$, and $B = 0.5$, respectively.