Problem 24

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

\[ 2 \sin y \sin x \cos x + \cos y \sin^2 x \frac{dy}{dx} = 0 \]

Solution

Method Using Separation of Variables

Divide both sides by \( \sin x \)

\[ 2 \sin y \cos x + \cos y \sin x \frac{dy}{dx} = 0 \]

and solve for \( \frac{dy}{dx} \).

\[ \frac{dy}{dx} = -\frac{2 \sin y \cos x}{\cos y \sin x} = -\frac{\sin y}{\cos y} \cdot \frac{2 \cos x}{\sin x} \]

Because the ODE is of the form \( y' = f(x)g(y) \), it can be solved by separating variables.

\[ \frac{\cos y}{\sin y} \frac{dy}{dx} = -2 \frac{\cos x}{\sin x} dx \]

Integrate both sides.

\[ \int \frac{\cos y}{\sin y} dy = \int -2 \frac{\cos x}{\sin x} dx \]

\[ \ln |\sin y| = -2 \ln |\sin x| + C \]

Isolate \( C \) on the right side.

\[ \ln |\sin y| + 2 \ln |\sin x| = C \]

\[ \ln |\sin y| + \ln \sin^2 x = C \]

\[ \ln |\sin y| \sin^2 x = C \]

Exponentiate both sides.

\[ |\sin y| \sin^2 x = e^C \]

Introduce \( \pm \) on the right side to remove the absolute value sign.

\[ \sin y \sin^2 x = \pm e^C \]

Therefore, using a new constant \( A \) for the right side,

\[ \sin y \sin^2 x = A. \]
Method Using an Integrating Factor

\[ 2 \sin y \sin x \cos x + \cos y \sin^2 x \frac{dy}{dx} = 0 \]  

(1)

This ODE is exact because

\[ \frac{\partial}{\partial y} (2 \sin y \sin x \cos x) = \frac{\partial}{\partial x} (\cos y \sin^2 x) = 2 \cos y \sin x \cos x. \]

That means there exists a potential function \( \psi = \psi(x, y) \) that satisfies

\[ \frac{\partial \psi}{\partial x} = 2 \sin y \sin x \cos x \]

(2)

\[ \frac{\partial \psi}{\partial y} = \cos y \sin^2 x. \]  

(3)

Integrate both sides of equation (3) partially with respect to \( y \) to get \( \psi \).

\[ \psi(x, y) = \sin y \sin^2 x + f(x) \]

Here \( f(x) \) is an arbitrary function of \( x \). Differentiate both sides with respect to \( x \).

\[ \psi_y(x, y) = 2 \sin y \sin x \cos x + f'(x) \]

Comparing this to equation (2), we see that \( f'(x) = 0 \) \( \rightarrow \) \( f(x) = 0 \).

As a result, a potential function is

\[ \psi(x, y) = \sin y \sin^2 x. \]

Notice that by substituting equations (2) and (3), equation (1) can be written as

\[ \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0. \]

(4)

Recall that the differential of \( \psi(x, y) \) is defined as

\[ d\psi = \frac{\partial \psi}{\partial x} \, dx + \frac{\partial \psi}{\partial y} \, dy. \]

Dividing both sides by \( dx \), we obtain the fundamental relationship between the total derivative of \( \psi \) and its partial derivatives.

\[ \frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} \]

With it, equation (4) becomes

\[ \frac{d\psi}{dx} = 0. \]

Integrate both sides with respect to \( x \).

\[ \psi(x, y) = C_1 \]

Therefore,

\[ \sin y \sin^2 x = C_1. \]
The Easy Way

\[ 2 \sin y \sin x \cos x + \cos y \sin^2 x \frac{dy}{dx} = 0 \]

Notice that the ODE can be written as
\[ \sin y \frac{d}{dx}(\sin^2 x) + \frac{d}{dy}(\sin y) \sin^2 x \frac{dy}{dx} = 0 \]
\[ \frac{\partial}{\partial x}(\sin y \sin^2 x) + \frac{\partial}{\partial y}(\sin y \sin^2 x) \frac{dy}{dx} = 0 \]
\[ \frac{d}{dx}(\sin y \sin^2 x) = 0. \]
Therefore, integrating both sides with respect to \(x\),
\[ \sin y \sin^2 x = B. \]

This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are \(B = -0.5\), \(B = -0.1\), \(B = -0.01\), \(B = 0.01\), \(B = 0.1\), and \(B = 0.5\), respectively.