Problem 25

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\left(2\frac{x}{y} - \frac{y}{x^2+y^2}\right) + \left(\frac{x}{x^2+y^2} - \frac{x^2}{y^2}\right)\frac{dy}{dx} = 0$$

Solution

This ODE is exact because

$$\frac{\partial}{\partial y}\left(2\frac{x}{y} - \frac{y}{x^2+y^2}\right) = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} - \frac{x^2}{y^2}\right) = \frac{y^2-x^2}{(x^2+y^2)^2} - 2\frac{x}{y}.$$

That means there exists a potential function $\psi = \psi(x, y)$ that satisfies

$$\frac{\partial \psi}{\partial x} = 2\frac{x}{y} - \frac{y}{x^2+y^2} \tag{1}$$

$$\frac{\partial \psi}{\partial y} = \frac{x}{x^2+y^2} - \frac{x^2}{y^2}. \tag{2}$$

Integrate both sides of equation (2) partially with respect to y to get ψ.

$$\psi(x, y) = \tan^{-1}\frac{y}{x} + \frac{x^2}{y} + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\begin{aligned}\psi_x(x, y) &= \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) + 2\frac{x}{y} + f'(x) \\ &= -\frac{y}{x^2+y^2} + 2\frac{x}{y} + f'(x)\end{aligned}$$

Comparing this to equation (1), we see that

$$f'(x) = 0 \quad \rightarrow \quad f(x) = 0.$$

As a result, a potential function is

$$\psi(x, y) = \tan^{-1}\frac{y}{x} + \frac{x^2}{y}.$$

Notice that by substituting equations (1) and (2), the original ODE can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{3}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (3) becomes

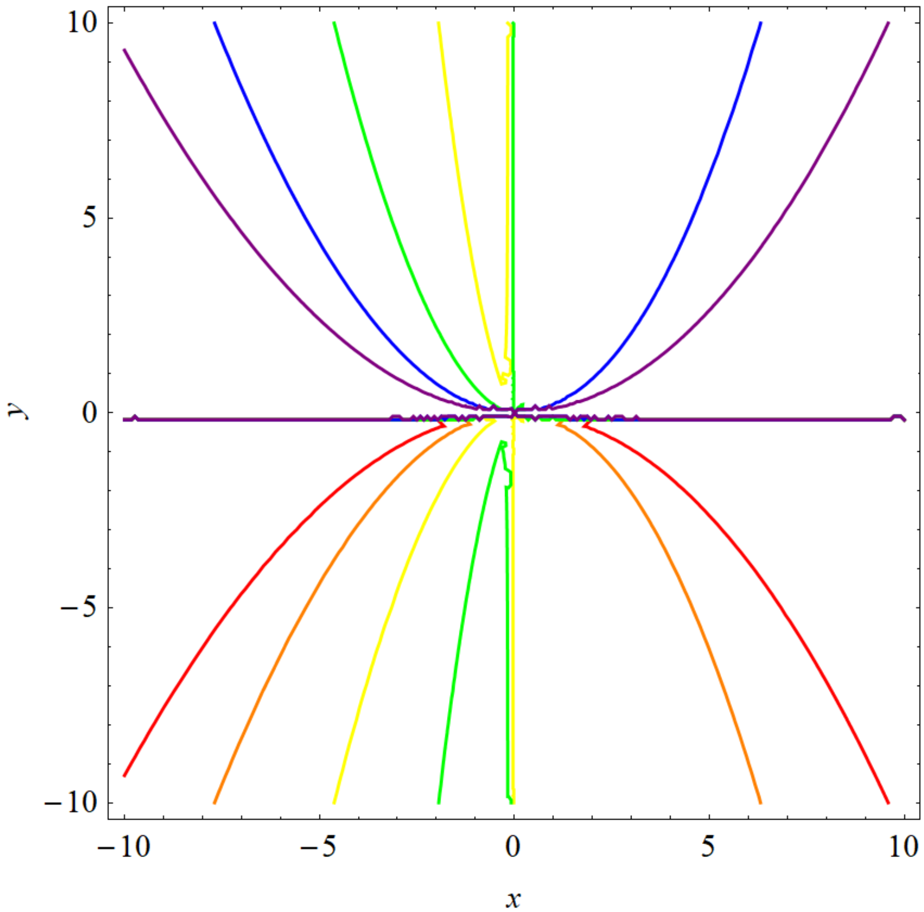
$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

$$\psi(x, y) = C$$

Therefore,

$$\tan^{-1}\frac{y}{x} + \frac{x^2}{y} = C.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.