Problem 25

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

\[
\left(\frac{2x}{y} - \frac{y}{x^2 + y^2}\right) + \left(\frac{x}{x^2 + y^2} - \frac{x^2}{y^2}\right) \frac{dy}{dx} = 0
\]

Solution

This ODE is exact because

\[
\frac{\partial}{\partial y} \left(\frac{2x}{y} - \frac{y}{x^2 + y^2}\right) = \frac{\partial}{\partial x} \left(\frac{x}{x^2 + y^2} - \frac{x^2}{y^2}\right) = \frac{y^2 - x^2}{(x^2 + y^2)^2} - 2\frac{x}{y}.
\]

That means there exists a potential function \( \psi = \psi(x, y) \) that satisfies

\[
\begin{align*}
\frac{\partial \psi}{\partial x} &= \frac{2x}{y} - \frac{y}{x^2 + y^2} \\
\frac{\partial \psi}{\partial y} &= \frac{x}{x^2 + y^2} - \frac{x^2}{y^2}.
\end{align*}
\]

Integrate both sides of equation (2) partially with respect to \( y \) to get \( \psi \).

\[
\psi(x, y) = \tan^{-1} \frac{y}{x} + \frac{x^2}{y} + f(x)
\]

Here \( f(x) \) is an arbitrary function of \( x \). Differentiate both sides with respect to \( x \).

\[
\psi_x(x, y) = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \left( -\frac{y}{x^2} \right) + 2\frac{x}{y} + f'(x) = -\frac{y}{x^2 + y^2} + 2\frac{x}{y} + f'(x)
\]

Comparing this to equation (1), we see that

\[
f'(x) = 0 \quad \rightarrow \quad f(x) = 0.
\]

As a result, a potential function is

\[
\psi(x, y) = \tan^{-1} \frac{y}{x} + \frac{x^2}{y}.
\]

Notice that by substituting equations (1) and (2), the original ODE can be written as

\[
\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0.
\]

Recall that the differential of \( \psi(x, y) \) is defined as

\[
d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.
\]

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Dividing both sides by $dx$, we obtain the fundamental relationship between the total derivative of $\psi$ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}$$

With it, equation (3) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to $x$.

$$\psi(x, y) = C$$

Therefore,

$$\tan^{-1} \frac{y}{x} + \frac{x^2}{y} = C.$$