Problem 28

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$(2y + 3x) = -x\frac{dy}{dx}$$

Solution

Method Using an Integrating Factor I

Bring the terms with dy/dx and y to the left side and functions of x to the right side.

$$x\frac{dy}{dx} + 2y = -3x$$

and divide both sides by x.

$$\frac{dy}{dx} + \frac{2}{x}y = -3$$

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor I.

$$I = \exp\left(\int^x \frac{2}{s}ds\right) = e^{2\ln x} = e^{\ln x^2} = x^2$$

Proceed with the multiplication.

$$x^2\frac{dy}{dx} + 2xy = -3x^2$$

The left side can be written as $d/dx(Iy)$ by the chain rule.

$$\frac{d}{dx}(x^2y) = -3x^2$$

Integrate both sides with respect to x.

$$x^2y = -x^3 + C$$

Therefore,

$$y(x) = -x + \frac{C}{x^2}.$$

Method Using an Integrating Factor II

$$(2y+3x) = -x\frac{dy}{dx}$$

Write the ODE as $M(x,y)+N(x,y)y'=0$.

$$(2y+3x)+x\frac{dy}{dx}=0 \tag{1}$$

This ODE is not exact at the moment because

$$\frac{\partial}{\partial y}(2y+3x)=2\neq\frac{\partial}{\partial x}(x)=1.$$

To solve it, we seek an integrating factor $\mu=\mu(x,y)$ such that when both sides are multiplied by it, the ODE becomes exact.

$$(2y+3x)\mu+x\mu\frac{dy}{dx}=0$$

Since the ODE is exact now,

$$\frac{\partial}{\partial y}[(2y+3x)\mu]=\frac{\partial}{\partial x}(x\mu).$$

Expand both sides.

$$2\mu+(2y+3x)\frac{\partial\mu}{\partial y}=\mu+x\frac{\partial\mu}{\partial x}$$

Assume that μ is only dependent on x: $\mu=\mu(x)$.

$$2\mu=\mu+x\frac{d\mu}{dx}$$

$$x\frac{d\mu}{dx}=\mu$$

Solve this ODE by separating variables.

$$\frac{d\mu}{\mu}=\frac{dx}{x}$$

Integrate both sides.

$$\ln\mu=\ln x+C_1$$

Exponentiate both sides.

$$\mu=xe^{C_1}$$

Taking e^{C_1} to be 1, an integrating factor is

$$\mu=x.$$

Multiply both sides of equation (1) by x.

$$(2xy+3x^2)+x^2\frac{dy}{dx}=0 \tag{2}$$

Because it's exact now, there exists a potential function $\psi=\psi(x,y)$ that satisfies

$$\frac{\partial\psi}{\partial x}=2xy+3x^2 \tag{3}$$

$$\frac{\partial\psi}{\partial y}=x^2. \tag{4}$$

Integrate both sides of equation (4) partially with respect to y to get ψ.

$$\psi(x, y) = x^2 y + f(x)$$

Here $f(x)$ is an arbitrary function of x. Differentiate both sides with respect to x.

$$\psi_x(x, y) = 2xy + f'(x)$$

Comparing this to equation (3), we see that

$$f'(x) = 3x^2 \quad \rightarrow \quad f(x) = x^3.$$

Consequently, a potential function is

$$\psi(x, y) = x^2 y + x^3.$$

Notice that by substituting equations (3) and (4), equation (2) can be written as

$$\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = 0. \tag{5}$$

Recall that the differential of $\psi(x, y)$ is defined as

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy.$$

Dividing both sides by dx, we obtain the fundamental relationship between the total derivative of ψ and its partial derivatives.

$$\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx}$$

With it, equation (5) becomes

$$\frac{d\psi}{dx} = 0.$$

Integrate both sides with respect to x.

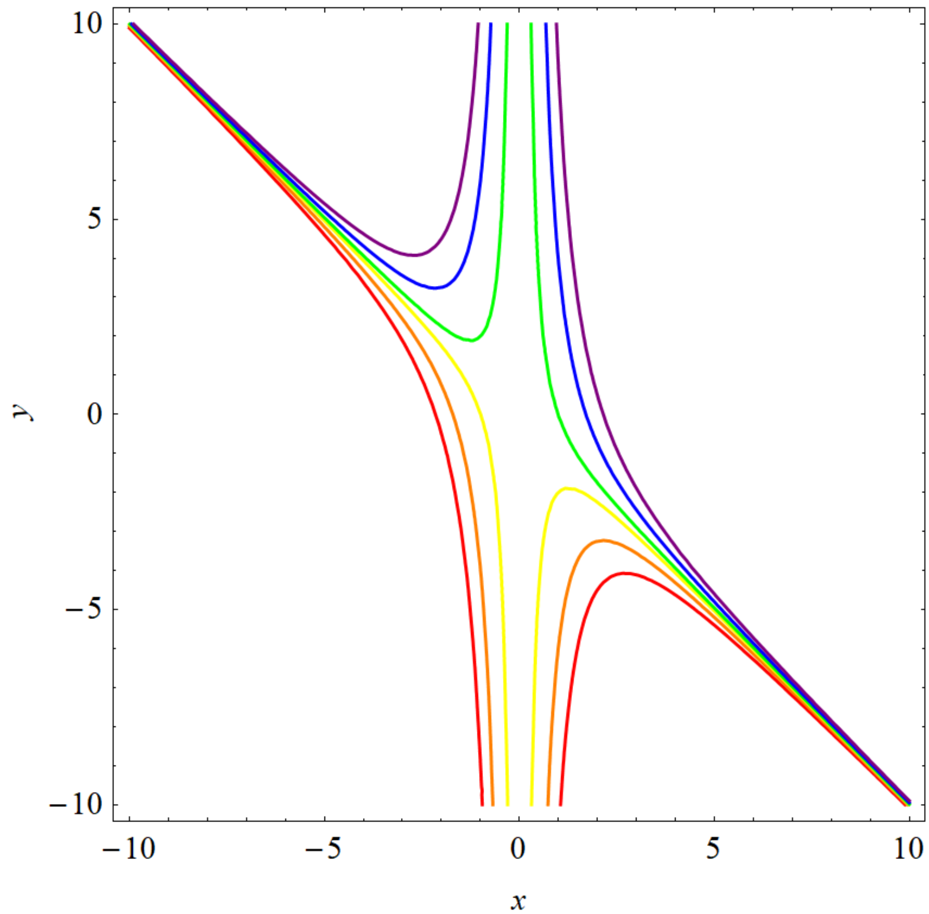
$$\psi(x, y) = C_2$$

Therefore,

$$x^2 y + x^3 = C_2,$$

or solving for y explicitly,

$$y(x) = -x + \frac{C_2}{x^2}.$$



This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.