Problem 28

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

\[(2y + 3x) = -x \frac{dy}{dx}\]

Solution

Method Using an Integrating Factor I

Bring the terms with \(dy/dx\) and \(y\) to the left side and functions of \(x\) to the right side.

\[x \frac{dy}{dx} + 2y = -3x\]

and divide both sides by \(x\).

\[\frac{dy}{dx} + \frac{2}{x}y = -3\]

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor \(I\).

\[I = \exp\left(\int \frac{2}{s} ds\right) = e^{2 \ln x} = e^{\ln x^2} = x^2\]

Proceed with the multiplication.

\[x^2 \frac{dy}{dx} + 2xy = -3x^2\]

The left side can be written as \(d/dx(Iy)\) by the chain rule.

\[\frac{d}{dx}(x^2y) = -3x^2\]

Integrate both sides with respect to \(x\).

\[x^2y = -x^3 + C\]

Therefore,

\[y(x) = -x + \frac{C}{x^2} \cdot\]
Method Using an Integrating Factor II

\[(2y + 3x) = -x \frac{dy}{dx}\]

Write the ODE as \(M(x, y) + N(x, y)y' = 0\).

\[(2y + 3x) + x \frac{dy}{dx} = 0 \quad (1)\]

This ODE is not exact at the moment because

\[\frac{\partial}{\partial y}(2y + 3x) = 2 \neq \frac{\partial}{\partial x}(x) = 1.\]

To solve it, we seek an integrating factor \(\mu = \mu(x, y)\) such that when both sides are multiplied by it, the ODE becomes exact.

\[(2y + 3x)\mu + x\mu \frac{dy}{dx} = 0\]

Since the ODE is exact now,

\[\frac{\partial}{\partial y}[(2y + 3x)\mu] = \frac{\partial}{\partial x}(x\mu).\]

Expand both sides.

\[2\mu + (2y + 3x) \frac{\partial \mu}{\partial y} = \mu + x \frac{\partial \mu}{\partial x}\]

Assume that \(\mu\) is only dependent on \(x\): \(\mu = \mu(x)\).

\[2\mu = \mu + x \frac{d\mu}{dx}\]

\[x \frac{d\mu}{dx} = \mu\]

Solve this ODE by separating variables.

\[\frac{d\mu}{\mu} = \frac{dx}{x}\]

Integrate both sides.

\[\ln \mu = \ln x + C_1\]

Exponentiate both sides.

\[\mu = xe^{C_1}\]

Taking \(e^{C_1}\) to be 1, an integrating factor is

\[\mu = x.\]

Multiply both sides of equation (1) by \(x\).

\[(2xy + 3x^2) + x^2 \frac{dy}{dx} = 0 \quad (2)\]

Because it’s exact now, there exists a potential function \(\psi = \psi(x, y)\) that satisfies

\[\frac{\partial \psi}{\partial x} = 2xy + 3x^2 \quad (3)\]

\[\frac{\partial \psi}{\partial y} = x^2, \quad (4)\]
Integrate both sides of equation (4) partially with respect to \( y \) to get \( \psi \).

\[
\psi(x, y) = x^2y + f(x)
\]

Here \( f(x) \) is an arbitrary function of \( x \). Differentiate both sides with respect to \( x \).

\[
\psi_x(x, y) = 2xy + f'(x)
\]

Comparing this to equation (3), we see that

\[
f'(x) = 3x^2 \quad \Rightarrow \quad f(x) = x^3.
\]

Consequently, a potential function is

\[
\psi(x, y) = x^2y + x^3.
\]

Notice that by substituting equations (3) and (4), equation (2) can be written as

\[
\frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx} = 0.
\]

(5)

Recall that the differential of \( \psi(x, y) \) is defined as

\[
d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.
\]

Dividing both sides by \( dx \), we obtain the fundamental relationship between the total derivative of \( \psi \) and its partial derivatives.

\[
\frac{d\psi}{dx} = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \frac{dy}{dx}
\]

With it, equation (5) becomes

\[
\frac{d\psi}{dx} = 0.
\]

Integrate both sides with respect to \( x \).

\[
\psi(x, y) = C_2
\]

Therefore,

\[
x^2y + x^3 = C_2,
\]

or solving for \( y \) explicitly,

\[
y(x) = -x + \frac{C_2}{x^2}.
\]
This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -10$, $C = -5$, $C = -1$, $C = 1$, $C = 5$, and $C = 10$, respectively.