# Problem 29

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

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### Solution

$$\frac{dy}{dx} = \frac{1+\frac{y}{x}}{1-\frac{y}{x}}$$

Make the substitution $u = y/x$.

$$\frac{dy}{dx} = \frac{1+u}{1-u}$$

Differentiate both sides of the substitution with respect to $x$ to find what $dy/dx$ is in terms of this new variable.

$$\frac{du}{dx} = \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} \quad \rightarrow \quad x\frac{du}{dx} = \frac{dy}{dx} - \frac{y}{x} \quad \rightarrow \quad \frac{dy}{dx} = x\frac{du}{dx} + \frac{y}{x} = x\frac{du}{dx} + u$$

Consequently, the ODE that $u$ satisfies is

$$x\frac{du}{dx} + u = \frac{1+u}{1-u}$$

$$\begin{aligned} x\frac{du}{dx} &= \frac{1+u}{1-u} - u \\ &= \frac{1+u-u(1-u)}{1-u} \\ &= \frac{1+u^2}{1-u}, \end{aligned}$$

which can be solved by separating variables.

$$\frac{1-u}{1+u^2}\,du = \frac{dx}{x}$$

Integrate both sides.

$$\int \frac{1-u}{1+u^2}\,du = \ln|x| + C$$

$$\int \frac{du}{1+u^2} - \int \frac{u}{1+u^2}\,du = \ln|x| + C$$

$$\tan^{-1} u - \frac{1}{2}\ln(1+u^2) = \ln|x| + C$$

Now that the integration is done, change back to $y$.

$$\tan^{-1}\frac{y}{x} - \frac{1}{2}\ln\left(1+\frac{y^2}{x^2}\right) = \ln|x| + C$$

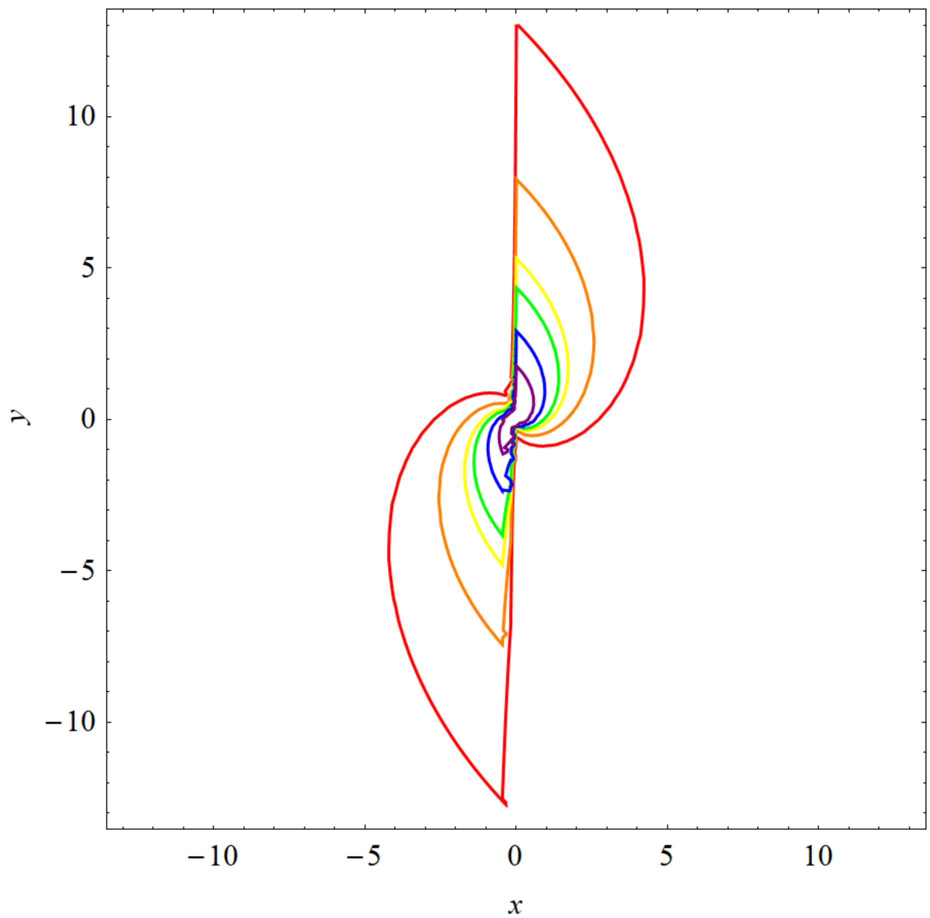
Bring $\ln|x|$ to the left side and bring the 1/2 to the argument's exponent.

$$\tan^{-1}\frac{y}{x} - \ln\sqrt{1+\frac{y^2}{x^2}} - \ln|x| = C$$

$$\tan^{-1}\frac{y}{x} - \ln|x|\sqrt{1+\frac{y^2}{x^2}} = C$$

Therefore,

$$\tan^{-1}\frac{y}{x} - \ln\sqrt{x^2+y^2} = C.$$




This figure illustrates several solutions of the family. In red, orange, yellow, green, blue, and purple are $C = -1$, $C = -0.5$, $C = -0.1$, $C = 0.1$, $C = 0.5$, and $C = 1$, respectively.