Problem 32

In each of Problems 1 through 32, solve the given differential equation. If an initial condition is given, also find the solution that satisfies it.

\[ xy' + y - y^2 e^{2x} = 0, \quad y(1) = 2 \]

Solution

This is a Bernoulli equation. Bring the third term to the right side

\[ xy' + y = y^2 e^{2x} \]

and divide both sides by \( xy^2 \).

\[ y^{-2} y' + \frac{1}{x} y^{-1} = \frac{e^{2x}}{x} \]

Make the substitution \( u = y^{-1} \). Differentiate both sides of it with respect to \( x \) to write \( y' \) in terms of this new variable.

\[ \frac{du}{dx} = -y^{-2} \cdot \frac{dy}{dx} \rightarrow y^{-2} y' = -\frac{du}{dx} \]

Consequently, the ODE that \( u \) satisfies is

\[ -\frac{du}{dx} + \frac{1}{x} u = \frac{e^{2x}}{x} \]

\[ \frac{d}{dx} \left( \frac{u}{x} \right) = -\frac{e^{2x}}{x^2} \]

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor \( I \).

\[ I = \exp \left( \int x \frac{1}{s} ds \right) = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} \]

Proceed with the multiplication.

\[ \frac{1}{x} \frac{du}{dx} - \frac{1}{x^2} u = -\frac{e^{2x}}{x^2} \]

The left side can be written as \( d/dx (Iu) \) by the chain rule.

\[ \frac{d}{dx} \left( \frac{u}{x} \right) = -\frac{e^{2x}}{x^2} \]

Integrate both sides with respect to \( x \).

\[ \frac{u}{x} = -\int x \frac{e^{2s}}{s^2} ds + C \]

The lower limit of integration is arbitrary since \( C \) is as well; \( C \) will be adjusted to account for whatever choice we make. Here it will be set to 1 because of the given boundary condition.

\[ \frac{u}{x} = -\int_1^x \frac{e^{2s}}{s^2} ds + C \]

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Now that the integration is done, change back to $y$.

$$\frac{1}{xy} = -\int_1^x \frac{e^{2s}}{s^2} \, ds + C$$

Apply the boundary condition $y(1) = 2$ to determine $C$.

$$\frac{1}{(1)(2)} = C \quad \rightarrow \quad C = \frac{1}{2}$$

The previous equation becomes

$$\frac{1}{xy} = -\int_1^x \frac{e^{2s}}{s^2} \, ds + \frac{1}{2}$$

$$\frac{1}{y} = -x \int_1^x \frac{e^{2s}}{s^2} \, ds + \frac{x}{2}$$

Therefore,

$$y(x) = \frac{1}{-x \int_1^x \frac{e^{2s}}{s^2} \, ds + \frac{x}{2}}$$

This figure illustrates the solution to the ODE in the $xy$-plane. Note that the solution is only valid along the curve that passes through the point $(1, 2)$. 