Problem 33

Riccati Equations. The equation

$$\frac{dy}{dt} = q_1(t) + q_2(t)y + q_3(t)y^2$$

is known as a Riccati[23] equation. Suppose that some particular solution y_1 of this equation is known. A more general solution containing one arbitrary constant can be obtained through the substitution

$$y = y_1(t) + \frac{1}{v(t)}.$$

Show that $v(t)$ satisfies the first order *linear* equation

$$\frac{dv}{dt} = -(q_2 + 2q_3y_1)v - q_3.$$

Note that $v(t)$ will contain a single arbitrary constant.

Solution

Differentiate both sides of the substitution with respect to t to find what dy/dt is in terms of the new variable.

$$\frac{dy}{dt} = \frac{dy_1}{dt} - \frac{1}{[v(t)]^2}\frac{dv}{dt}$$

Substitute this and the expression for y into the ODE.

$$\frac{dy_1}{dt} - \frac{1}{[v(t)]^2}\frac{dv}{dt} = q_1(t) + q_2(t)\left[y_1(t) + \frac{1}{v(t)}\right] + q_3(t)\left[y_1(t) + \frac{1}{v(t)}\right]^2$$

Substitute $q_1(t) + q_2(t)y_1 + q_3(t)y_1^2$ for dy_1/dt and expand the right side.

$$q_1 + q_2y_1 + q_3y_1^2 - \frac{1}{v^2}\frac{dv}{dt} = q_1 + q_2y_1 + \frac{q_2}{v} + q_3\left(y_1^2 + 2\frac{y_1}{v} + \frac{1}{v^2}\right)$$

$$q_1 + q_2y_1 + q_3y_1^2 - \frac{1}{v^2}\frac{dv}{dt} = q_1 + q_2y_1 + \frac{q_2}{v} + q_3y_1^2 + 2\frac{q_3y_1}{v} + \frac{q_3}{v^2}$$

Cancel terms common on both sides.

$$-\frac{1}{v^2}\frac{dv}{dt} = \frac{q_2}{v} + 2\frac{q_3y_1}{v} + \frac{q_3}{v^2}$$

Multiply both sides by $-v^2$.

$$\frac{dv}{dt} = -q_2v - 2q_3y_1v - q_3$$

Therefore, v satisfies the following ODE.

$$\frac{dv}{dt} = -(q_2 + 2q_3y_1)v - q_3$$

[23] Riccati equations are named for Jacopo Francesco Riccati (1676–1754), a Venetian nobleman, who declined university appointments in Italy, Austria, and Russia to pursue his mathematical studies privately at home. Riccati studied these equations extensively; however, it was Euler (in 1760) who discovered the result stated in this problem.