Problem 33

Riccati Equations. The equation
\[
\frac{dy}{dt} = q_1(t) + q_2(t)y + q_3(t)y^2
\]
is known as a Riccati\textsuperscript{23} equation. Suppose that some particular solution \(y_1\) of this equation is known. A more general solution containing one arbitrary constant can be obtained through the substitution
\[
y = y_1(t) + \frac{1}{v(t)}.
\]

Show that \(v(t)\) satisfies the first order linear equation
\[
\frac{dv}{dt} = -(q_2 + 2q_3y_1)v - q_3.
\]

Note that \(v(t)\) will contain a single arbitrary constant.

Solution

Differentiate both sides of the substitution with respect to \(t\) to find what \(dy/dt\) is in terms of the new variable.
\[
\frac{dy}{dt} = \frac{dy_1}{dt} - \frac{1}{[v(t)]^2} \frac{dv}{dt}
\]

Substitute this and the expression for \(y\) into the ODE.
\[
\frac{dy_1}{dt} - \frac{1}{[v(t)]^2} \frac{dv}{dt} = q_1(t) + q_2(t) \left[ y_1(t) + \frac{1}{v(t)} \right] + q_3(t) \left[ y_1(t) + \frac{1}{v(t)} \right]^2
\]

Substitute \(q_1(t) + q_2(t)y_1 + q_3(t)y_1^2\) for \(dy_1/dt\) and expand the right side.
\[
q_1 + q_2y_1 + q_3y_1^2 - \frac{1}{v^2} \frac{dv}{dt} = q_1 + q_2y_1 + \frac{q_2}{v} + q_3 \left( y_1^2 + 2\frac{y_1}{v} + \frac{1}{v^2} \right)
\]
\[
q_1 + q_2y_1 + q_3y_1^2 - \frac{1}{v^2} \frac{dv}{dt} = q_1 + q_2y_1 + \frac{q_2}{v} + q_3y_1^2 + 2\frac{q_3y_1}{v} + \frac{q_3}{v^2}
\]

Cancel terms common on both sides.
\[
-\frac{1}{v^2} \frac{dv}{dt} = \frac{q_2}{v} + 2\frac{q_3y_1}{v} + \frac{q_3}{v^2}
\]

Multiply both sides by \(-v^2\).
\[
\frac{dv}{dt} = -q_2v - 2q_3y_1v - q_3
\]

Therefore, \(v\) satisfies the following ODE.
\[
\frac{dv}{dt} = -(q_2 + 2q_3y_1)v - q_3
\]

\textsuperscript{23}Riccati equations are named for Jacopo Francesco Riccati (1676–1754), a Venetian nobleman, who declined university appointments in Italy, Austria, and Russia to pursue his mathematical studies privately at home. Riccati studied these equations extensively; however, it was Euler (in 1760) who discovered the result stated in this problem.

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