Problem 34

Using the method of Problem 33 and the given particular solution, solve each of the following Riccati equations:

(a) \( y' = 1 + t^2 - 2ty + y^2; \quad y_1(t) = t \)  
(b) \( y' = -\frac{1}{t^2} - \frac{y}{t} + y^2; \quad y_1(t) = \frac{1}{t} \)

(c) \( \frac{dy}{dt} = \frac{2\cos^2 t - \sin^2 t + y^2}{2\cos t}; \quad y_1(t) = \sin t \)

Solution

Part (a)

Make the substitution

\[ y = t + \frac{1}{v(t)}. \]

Differentiate both sides of the substitution with respect to \( t \) to find what \( dy/dt \) is in terms of this new variable.

\[ \frac{dy}{dt} = 1 - \frac{1}{v(t)} \frac{dv}{dt} \]

Substitute these previous two expressions into the ODE.

\[ 1 - \frac{1}{v(t)^2} \frac{dv}{dt} = 1 + t^2 - 2t \left[ t + \frac{1}{v(t)} \right] + \left[ t + \frac{1}{v(t)} \right]^2 \]

Cancel 1 from both sides and expand the right side.

\[ -\frac{1}{v^2} \frac{dv}{dt} = t^2 - 2t^2 - \frac{2t}{v} + t^2 + \frac{2t}{v} + \frac{1}{v^2} \]

\[ -\frac{1}{v^2} \frac{dv}{dt} = \frac{1}{v^2} \]

Multiply both sides by \(-v^2\).

\[ \frac{dv}{dt} = -1 \]

Integrate both sides with respect to \( t \).

\[ v(t) = -t + C \]

Therefore, the general solution is

\[ y(t) = t + \frac{1}{-t + C}. \]
Part (b)

\[ y' = -\frac{1}{t^2} - \frac{y}{t} + y^2; \quad y_1(t) = \frac{1}{t} \]

Make the substitution

\[ y = \frac{1}{t} + \frac{1}{v(t)}. \]

Differentiate both sides of the substitution with respect to \( t \) to find what \( dy/dt \) is in terms of this new variable.

\[ \frac{dy}{dt} = -\frac{1}{t^2} - \frac{1}{v(t)^2} \frac{dv}{dt} \]

Substitute these previous two expressions into the ODE.

\[ -\frac{1}{t^2} - \frac{1}{v(t)^2} \frac{dv}{dt} = -\frac{1}{t^2} - \frac{1}{t} \left[ \frac{1}{t} + \frac{1}{v(t)} \right] + \left[ \frac{1}{t} + \frac{1}{v(t)} \right]^2 \]

Cancel \(-1/t^2\) from both sides and expand the right side.

\[ -\frac{1}{v^2} \frac{dv}{dt} = -\frac{1}{t^2} - \frac{1}{tv} + \frac{2}{tv} + \frac{1}{v^2} \]

\[ = \frac{1}{tv} + \frac{1}{v^2} \]

Multiply both sides by \(-v^2\).

\[ \frac{dv}{dt} = \frac{1}{tv} - 1 \]

\[ \frac{dv}{dt} + \frac{1}{tv} = -1 \]

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor \( I \).

\[ I = \exp \left( \int \frac{1}{s} \, ds \right) = e^{\ln t} = t \]

Proceed with the multiplication.

\[ t \frac{dv}{dt} + v = -t \]

The left side can be written as \( d/dt(Iv) \) by the chain rule.

\[ \frac{d}{dt}(tv) = -t \]

Integrate both sides with respect to \( t \).

\[ tv = -\frac{t^2}{2} + C \]

Divide both sides by \( t \).

\[ v(t) = -\frac{t^2}{2} + C \]

Therefore, the general solution is

\[ y(t) = \frac{1}{t} + \frac{t}{-\frac{t^2}{2} + C}. \]

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Part (c)

\[ \frac{dy}{dt} = \frac{2 \cos^2 t - \sin^2 t + y^2}{2 \cos t} ; \quad y_1(t) = \sin t \]

Make the substitution

\[ y = \sin t + \frac{1}{v(t)}. \]

Differentiate both sides of the substitution with respect to \( t \) to find what \( dy/dt \) is in terms of this new variable.

\[ \frac{dy}{dt} = \cos t - \frac{1}{v(t)^2} \frac{dv}{dt} \]

Substitute these previous two expressions into the ODE.

\[
\begin{align*}
\cos t - \frac{1}{v(t)^2} \frac{dv}{dt} &= \frac{2 \cos^2 t - \sin^2 t + \left[ \sin t + \frac{1}{v(t)} \right]^2}{2 \cos t} \\
&= \frac{2 \cos^2 t - \sin^2 t + \sin^2 t + 2 \sin t \left[ \frac{1}{v(t)} \right] + \frac{1}{v(t)^2}}{2 \cos t} \\
&= \frac{2 \cos^2 t + 2 \sin t \left[ \frac{1}{v(t)} \right] + \frac{1}{v(t)^2}}{2 \cos t} \\
&= \cos t + \frac{\sin t}{\cos t} \cdot \frac{1}{v(t)} + \frac{1}{2 \cos t} \cdot \frac{1}{v(t)^2} \\
\end{align*}
\]

Cancel \( \cos t \) from both sides.

\[ \frac{1}{v(t)^2} \frac{dv}{dt} = \sin t \cdot \frac{1}{\cos t} \cdot \frac{1}{v(t)} + \frac{1}{2 \cos t} \cdot \frac{1}{v(t)^2} \]

Multiply both sides by \(-v^2\).

\[ \frac{dv}{dt} = \frac{\sin t}{\cos t} \cdot v - \frac{1}{2 \cos t} \]

This is a first-order linear inhomogeneous ODE, so it can be solved by multiplying both sides by an integrating factor \( I \).

\[ I = \exp \left( \int \frac{\sin s}{\cos s} ds \right) = e^{-\ln \cos t} = e^{\ln \frac{1}{\cos t}} = \frac{1}{\cos t} \]

Proceed with the multiplication.

\[ \frac{1}{\cos t} \frac{dv}{dt} + \frac{\sin t}{\cos^2 t} \cdot v = -\frac{1}{2 \cos^2 t} \]

The left side can be written as \( d/dt(Iv) \) by the chain rule.

\[ \frac{d}{dt} \left( \frac{v}{\cos t} \right) = -\frac{1}{2 \cos^2 t} \]
Integrate both sides with respect to $t$.

$$\frac{v}{\cos t} = \int -\frac{1}{2 \cos^2 t} \, dt$$

$$= -\frac{1}{2} \int \sec^2 t \, dt$$

$$= -\frac{1}{2} \tan t + C$$

$$= -\frac{1}{2} \sin t + C$$

Multiply both sides by $\cos t$.

$$v(t) = -\frac{1}{2} \sin t + C \cos t$$

Therefore, the general solution is

$$y(t) = \sin t + \frac{1}{-\frac{1}{2} \sin t + C \cos t}.$$