Problem 48

In each of Problems 48 through 51, solve the given initial value problem using the methods of Problems 36 through 47.

\[ y' y'' = 2, \quad y(0) = 1, \quad y'(0) = 2 \]

Solution

Rewrite the left side of the ODE by using the chain rule.

\[ \frac{1}{2}(2y')y'' = 2 \]

\[ \frac{1}{2} \frac{d}{dt}[(y')^2] = 2 \]

Multiply both sides by 2.

\[ \frac{d}{dt}[(y')^2] = 4 \]

Integrate both sides with respect to \( t \).

\[ (y')^2 = 4t + C_1 \]

Apply the second initial condition \( y'(0) = 2 \) here to determine \( C_1 \).

\[ (2)^2 = C_1 \quad \rightarrow \quad C_1 = 4 \]

So the previous equation becomes

\[ (y')^2 = 4t + 4. \]

Take the square root of both sides.

\[ \frac{dy}{dt} = \pm \sqrt{4t + 4} \]

The plus sign is chosen so that the second initial condition remains satisfied.

\[ \frac{dy}{dt} = \sqrt{4t + 4} \]

Integrate both sides with respect to \( t \) once more.

\[ y(t) = \int \sqrt{4s + 4} \, ds + C_2 \]

Make the following substitution.

\[ w = 4s + 4 \]

\[ dw = 4 \, ds \quad \rightarrow \quad \frac{dw}{4} = ds \]
As a result,

\[
y(t) = \int_{4t+4}^{4t+4} \sqrt{w} \, dw + C_2
\]

\[
= \frac{1}{4} \int_{4t+4}^{4t+4} w^{1/2} \, dw + C_2
\]

\[
= \frac{1}{4} \cdot \frac{2}{3} (4t + 4)^{3/2} + C_2
\]

\[
= \frac{1}{6} \cdot 4^{3/2} (t + 1)^{3/2} + C_2
\]

\[
= \frac{4}{3} (t + 1)^{3/2} + C_2.
\]

Apply the first initial condition \( y(0) = 1 \) to determine \( C_2 \).

\[
1 = \frac{4}{3} (1)^{3/2} + C_2 \quad \rightarrow \quad C_2 = -\frac{1}{3}
\]

Therefore,

\[
y(t) = \frac{4}{3} (t + 1)^{3/2} - \frac{1}{3}.
\]