Problem 10

In each of Problems 9 through 16, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as t increases.

$$y'' + 4y' + 3y = 0, \qquad y(0) = 2, \quad y'(0) = -1$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 4(re^{rt}) + 3(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 4r + 3 = 0$$

$$(r+3)(r+1) = 0$$

$$r = \{-3, -1\}$$

Two solutions to the ODE are $y = e^{-3t}$ and $y = e^{-t}$, so the general solution is

$$y(t) = C_1 e^{-3t} + C_2 e^{-t},$$

a linear combination of the two. Differentiate it once with respect to t.

$$y'(t) = -3C_1 e^{-3t} - C_2 e^{-t}$$

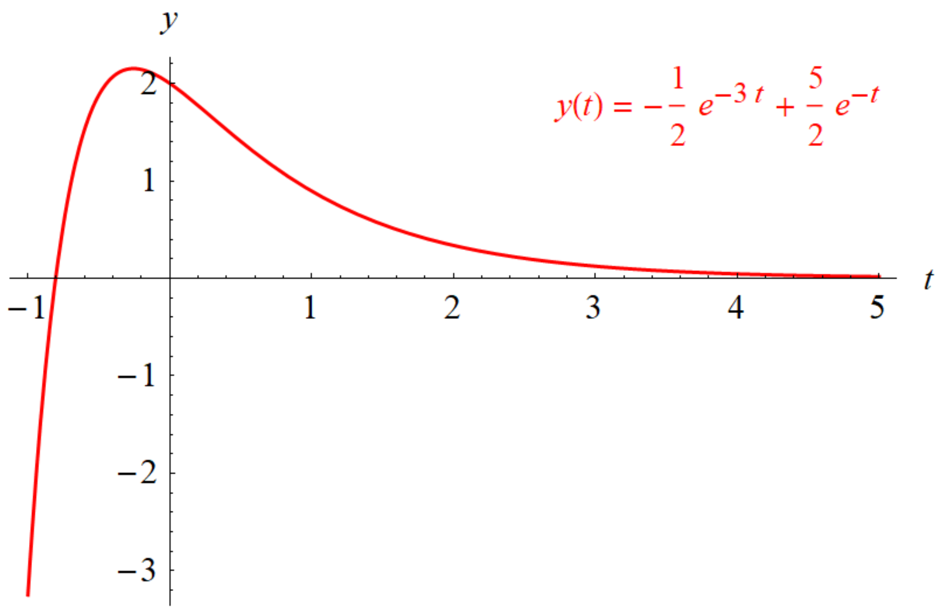
Apply the two initial conditions now to determine C_1 and C_2.

$$y(0) = C_1 + C_2 = 2$$
$$y'(0) = -3C_1 - C_2 = -1$$

Solving the system of equations yields $C_1 = -1/2$ and $C_2 = 5/2$. Therefore,

$$y(t) = -\frac{1}{2} e^{-3t} + \frac{5}{2} e^{-t}.$$

This solution converges to 0 as $t \to \infty$.



y
t
2
1
−1
−2
−3
−1
1
2
3
4
5
$y(t) = -\frac{1}{2}e^{-3t} + \frac{5}{2}e^{-t}$