Problem 11

In each of Problems 9 through 16, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior as \( t \) increases.

\[ 6y'' - 5y' + y = 0, \quad y(0) = 4, \quad y'(0) = 0 \]

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form \( y = e^{rt} \).

\[ y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt} \]

Substitute these expressions into the ODE.

\[ 6(r^2 e^{rt}) - 5(re^{rt}) + e^{rt} = 0 \]

Divide both sides by \( e^{rt} \).

\[ 6r^2 - 5r + 1 = 0 \]

\[ (3r - 1)(2r - 1) = 0 \]

\[ r = \left\{ \frac{1}{3}, \frac{1}{2} \right\} \]

Two solutions to the ODE are \( y = e^{t/3} \) and \( y = e^{t/2} \), so the general solution is

\[ y(t) = C_1 e^{t/3} + C_2 e^{t/2}, \]

a linear combination of the two. Differentiate it once with respect to \( t \).

\[ y'(t) = \frac{C_1}{3} e^{t/3} + \frac{C_2}{2} e^{t/2} \]

Apply the two initial conditions now to determine \( C_1 \) and \( C_2 \).

\[ y(0) = C_1 + C_2 = 4 \]

\[ y'(0) = \frac{C_1}{3} + \frac{C_2}{2} = 0 \]

Solving the system of equations yields \( C_1 = 12 \) and \( C_2 = -8 \). Therefore,

\[ y(t) = 12e^{t/3} - 8e^{t/2} \]

Take the limit of \( y(t) \) as \( t \to \infty \).

\[ \lim_{t \to \infty} y(t) = \lim_{t \to \infty} (12e^{t/3} - 8e^{t/2}) \]

\[ = \lim_{t \to \infty} 4e^{t/2}(3e^{-t/6} - 2) \]

\[ = \lim_{t \to \infty} 4e^{t/2}(-2) \]

\[ = \lim_{t \to \infty} -8e^{t/2} \]

This solution diverges to \(-\infty\) as \( t \to \infty \).
$y(t) = 12 e^{t/3} - 8 e^{t/2}$