Problem 20

Find the solution of the initial value problem

\[ 2y'' - 3y' + y = 0, \quad y(0) = 2, \quad y'(0) = \frac{1}{2}. \]

Then determine the maximum value of the solution and also find the point where the solution is zero.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form \( y = e^{rt} \).

\[ y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt} \]

Substitute these expressions into the ODE.

\[ 2(r^2 e^{rt}) - 3(re^{rt}) + e^{rt} = 0 \]

Divide both sides by \( e^{rt} \).

\[ 2r^2 - 3r + 1 = 0 \]

\[ (2r - 1)(r - 1) = 0 \]

\[ r = \left\{ \frac{1}{2}, 1 \right\} \]

Two solutions to the ODE are \( y = e^{t/2} \) and \( y = e^t \), so the general solution is

\[ y(t) = C_1 e^{t/2} + C_2 e^t, \]

a linear combination of the two. Differentiate it once with respect to \( t \).

\[ y'(t) = \frac{C_1}{2} e^{t/2} + C_2 e^t \]

Apply the two initial conditions now to determine \( C_1 \) and \( C_2 \).

\[ y(0) = C_1 + C_2 = 2 \]

\[ y'(0) = \frac{C_1}{2} + C_2 = \frac{1}{2} \]

Solving the system of equations yields \( C_1 = 3 \) and \( C_2 = -1 \). Therefore,

\[ y(t) = 3e^{t/2} - e^t. \]
The maximum value is found by solving \( y'(t) = 0 \) for \( t \), and the zero is found by solving \( y(t) = 0 \) for \( t \).

\[
\begin{align*}
y'(t) &= 0 \\
\frac{3}{2} e^{t/2} - e^t &= 0 \\
y(t) &= 0 \\
3 e^{t/2} - e^t &= 0
\end{align*}
\]

Divide both sides of each equation by \( e^{t/2} \).

\[
\begin{align*}
\frac{3}{2} - e^{t/2} &= 0 \\
e^{t/2} &= \frac{3}{2} \\
\ln e^{t/2} &= \ln \frac{3}{2} \\
t &= \ln \frac{3}{2} \\
t &= 2 \ln 3 \approx 2.20
\end{align*}
\]

Therefore, the maximum is

\[
y\left(2 \ln \frac{3}{2}\right) = \frac{9}{4},
\]

and the zero is at \((2 \ln 3, 0)\).