Problem 22

Solve the initial value problem $4y'' - y = 0$, $y(0) = 2$, $y'(0) = \beta$. Then find β so that the solution approaches zero as $t \to \infty$.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$4(r^2 e^{rt}) - e^{rt} = 0$$

Divide both sides by e^{rt}.

$$4r^2 - 1 = 0$$

$$(2r + 1)(2r - 1) = 0$$

$$r = \left\{-\frac{1}{2}, \frac{1}{2}\right\}$$

Two solutions to the ODE are $y = e^{-t/2}$ and $y = e^{t/2}$, so the general solution is

$$y(t) = C_1 e^{-t/2} + C_2 e^{t/2},$$

a linear combination of the two. Differentiate it once with respect to t.

$$y'(t) = -\frac{C_1}{2} e^{-t/2} + \frac{C_2}{2} e^{t/2}$$

Apply the two initial conditions now to determine C_1 and C_2.

$$y(0) = C_1 + C_2 = 2$$

$$y'(0) = -\frac{C_1}{2} + \frac{C_2}{2} = \beta$$

Solving the system of equations yields $C_1 = 1 - \beta$ and $C_2 = 1 + \beta$. Therefore,

$$y(t) = (1 - \beta)e^{-t/2} + (1 + \beta)e^{t/2}.$$

To prevent the solution from blowing up as $t \to \infty$, set $\beta = -1$.