Problem 25

Consider the initial value problem

\[ 2y'' + 3y' - 2y = 0, \quad y(0) = 1, \quad y'(0) = -\beta, \]

where \( \beta > 0 \).

(a) Solve the initial value problem.

(b) Plot the solution when \( \beta = 1 \). Find the coordinates \((t_0, y_0)\) of the minimum point of the solution in this case.

(c) Find the smallest value of \( \beta \) for which the solution has no minimum point.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form \( y = e^{rt} \).

\[
y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}
\]

Substitute these expressions into the ODE.

\[
2(r^2 e^{rt}) + 3(re^{rt}) - 2(e^{rt}) = 0
\]

Divide both sides by \( e^{rt} \).

\[
2r^2 + 3r - 2 = 0
\]

\[
(2r - 1)(r + 2) = 0
\]

\[
r = \left\{ -\frac{1}{2}, \frac{1}{2} \right\}
\]

Two solutions to the ODE are \( y = e^{-2t} \) and \( y = e^{t/2} \). Therefore, the general solution is

\[
y(t) = C_1 e^{-2t} + C_2 e^{t/2},
\]

a linear combination of the two. Differentiate it once with respect to \( t \).

\[
y'(t) = -2C_1 e^{-2t} + \frac{C_2}{2} e^{t/2}
\]

Apply the two initial conditions now to determine \( C_1 \) and \( C_2 \).

\[
y(0) = C_1 + C_2 = 1
\]

\[
y'(0) = -2C_1 + \frac{C_2}{2} = -\beta
\]

Solving this system of equations yields

\[
C_1 = \frac{1}{5}(1 + 2\beta)
\]

\[
C_2 = \frac{2}{5}(2 - \beta).
\]

Therefore,

\[
y(t) = \frac{1}{5}(1 + 2\beta)e^{-2t} + \frac{2}{5}(2 - \beta)e^{t/2}.
\]
In the case that $\beta = 1$, the solution becomes

$$y(t) = \frac{3}{5} e^{-2t} + \frac{2}{5} e^{t/2}.$$ 

To find where the minimum is, solve $y'(t) = 0$ for $t$.

$$y'(t) = -\frac{6}{5} e^{-2t} + \frac{1}{5} e^{t/2} = 0$$

Divide both sides by $e^{t/2}$.

$$-\frac{6}{5} e^{-5t/2} + \frac{1}{5} = 0$$

$$-6 e^{-5t/2} + 1 = 0$$

$$6 e^{-5t/2} = 1$$

$$e^{-5t/2} = \frac{1}{6}$$

$$-\frac{5t}{2} = \ln \frac{1}{6}$$

$$t = -\frac{2}{5} \ln \frac{1}{6}$$

$$= \frac{2}{5} \ln 6 \approx 0.717$$

Therefore, the minimum is at

$$y \left( \frac{2}{5} \ln 6 \right) = \frac{3^{1/5}}{5} \cdot 2^{4/5} + \frac{2 \cdot 6^{1/5}}{5} \approx 0.715.$$
Return to the original solution.

\[ y(t) = \frac{1}{5}(1 + 2\beta)e^{-2t} + \frac{2}{5}(2 - \beta)e^{t/2}. \]

Take a derivative with respect to \( t \).

\[ y'(t) = -\frac{2}{5}(1 + 2\beta)e^{-2t} + \frac{1}{5}(2 - \beta)e^{t/2}. \]

The graph of \( y(t) \) has a minimum if \( 0 < \beta < 2 \), so the smallest value of \( \beta \) for which there is no minimum is \( \beta = 2 \).