Problem 28

Consider the equation \( ay'' + by' + cy = 0 \), where \( a, b, \) and \( c \) are constants with \( a > 0 \). Find conditions on \( a, b, \) and \( c \) such that the roots of the characteristic equation are:

(a) real, different, and negative.

(b) real with opposite signs.

(c) real, different, and positive.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form \( y = e^{rt} \).

\[
y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2e^{rt}
\]

Substitute these expressions into the ODE.

\[
a(r^2e^{rt}) + b(re^{rt}) + c(e^{rt}) = 0
\]

Divide both sides by \( e^{rt} \).

\[
ar^2 + br + c = 0
\]

\[
r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

Part (a)

For the roots to be real, different and negative, \( b^2 - 4ac > 0 \) and \( b > 0 \) and \( c > 0 \).

Part (b)

For the roots to be real with opposite signs, \( c < 0 \).

Part (c)

For the roots to be real, different and positive, \( b^2 - 4ac > 0 \) and \( b < 0 \) and \( c > 0 \).

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