Problem 14

Verify that $y_1(t) = 1$ and $y_2(t) = t^{1/2}$ are two solutions of the differential equation $yy'' + (y')^2 = 0$ for $t > 0$. Then show that $y = c_1 + c_2 t^{1/2}$ is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 3.2.2.

Solution

Check that $y_1(t)$ is a solution.

$$y_1 y_1'' + (y_1')^2 \stackrel{?}{=} 0$$

$$(1)\frac{d^2}{dt^2}(1) + \left[\frac{d}{dt}(1)\right]^2 \stackrel{?}{=} 0$$

$$(1)(0) + [(0)]^2 \stackrel{?}{=} 0$$

$$0 = 0$$

Now check that $y_2(t)$ is a solution.

$$y_2 y_2'' + (y_2')^2 \stackrel{?}{=} 0$$

$$(t^{1/2})\frac{d^2}{dt^2}(t^{1/2}) + \left[\frac{d}{dt}(t^{1/2})\right]^2 \stackrel{?}{=} 0$$

$$(t^{1/2})\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)t^{-3/2} + \left[\left(\frac{1}{2}\right)t^{-1/2}\right]^2 \stackrel{?}{=} 0$$

$$-\frac{1}{4}t^{-1} + \frac{1}{4}t^{-1} \stackrel{?}{=} 0$$

$$0 = 0$$

Now check that $y(t)$ is not a solution.

$$yy'' + (y')^2 \stackrel{?}{=} 0$$

$$(c_1 + c_2 t^{1/2})\frac{d^2}{dt^2}(c_1 + c_2 t^{1/2}) + \left[\frac{d}{dt}(c_1 + c_2 t^{1/2})\right]^2 \stackrel{?}{=} 0$$

$$(c_1 + c_2 t^{1/2})\left[c_2\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)t^{-3/2}\right] + \left[c_2\left(\frac{1}{2}\right)t^{-1/2}\right]^2 \stackrel{?}{=} 0$$

$$-\frac{c_1 c_2}{4}t^{-3/2} - \frac{c_2^2}{4}t^{-1} + \frac{c_2^2}{4}t^{-1} \stackrel{?}{=} 0$$

$$-\frac{c_1 c_2}{4}t^{-3/2} \neq 0$$

This result does not contradict Theorem 3.2.2 because the theorem assumes the ODE is of the form $y'' + p(t)y' + q(t)y = 0$. The ODE dealt with in this problem is $yy'' + (y')^2 = 0$, a different one, so Theorem 3.2.2 does not apply.