Problem 14

Verify that \( y_1(t) = 1 \) and \( y_2(t) = t^{1/2} \) are two solutions of the differential equation \( yy'' + (y')^2 = 0 \) for \( t > 0 \). Then show that \( y = c_1 + c_2 t^{1/2} \) is not, in general, a solution of this equation. Explain why this result does not contradict Theorem 3.2.2.

Solution

Check that \( y_1(t) \) is a solution.

\[
yy'' + (y')^2 = 0
\]

\[
(1) \frac{d^2}{dt^2}(1) + \left[ \frac{d}{dt}(1) \right]^2 = 0
\]

\[
(1)(0) + [(0)]^2 = 0
\]

\[
0 = 0
\]

Now check that \( y_2(t) \) is a solution.

\[
yy'' + (y')^2 = 0
\]

\[
(t^{1/2}) \frac{d^2}{dt^2}(t^{1/2}) + \left[ \frac{d}{dt}(t^{1/2}) \right]^2 = 0
\]

\[
(t^{1/2}) \left( \frac{1}{2} \right) \left( -\frac{1}{2} \right) t^{-3/2} + \left[ \left( \frac{1}{2} \right) t^{-1/2} \right]^2 = 0
\]

\[
-\frac{1}{4} t^{-1} + \frac{1}{4} t^{-1} = 0
\]

\[
0 = 0
\]

Now check that \( y(t) \) is not a solution.

\[
yy'' + (y')^2 = 0
\]

\[
(c_1 + c_2 t^{1/2}) \frac{d^2}{dt^2}(c_1 + c_2 t^{1/2}) + \left[ \frac{d}{dt}(c_1 + c_2 t^{1/2}) \right]^2 = 0
\]

\[
(c_1 + c_2 t^{1/2}) \left[ c_2 \left( \frac{1}{2} \right) \left( -\frac{1}{2} \right) t^{-3/2} \right] + \left[ c_2 \left( \frac{1}{2} \right) t^{-1/2} \right]^2 = 0
\]

\[
-\frac{c_1 c_2}{4} t^{-3/2} - \frac{c_2^2}{4} t^{-1} + \frac{c_2^2}{4} t^{-1} = 0
\]

\[
-\frac{c_1 c_2}{4} t^{-3/2} \neq 0
\]

This result does not contradict Theorem 3.2.2 because the theorem assumes the ODE is of the form \( y'' + p(t)y' + q(t)y = 0 \). The ODE dealt with in this problem is \( yy'' + (y')^2 = 0 \), a different one, so Theorem 3.2.2 does not apply.