Problem 15

Show that if \( y = \phi(t) \) is a solution of the differential equation \( y'' + p(t)y' + q(t)y = g(t) \), where \( g(t) \) is not always zero, then \( y = c\phi(t) \), where \( c \) is any constant other than 1, is not a solution. Explain why this result does not contradict the remark following Theorem 3.2.2.

Solution

Suppose that \( \phi(t) \) is a solution of the ODE. Then

\[
\phi'' + p(t)\phi' + q(t)\phi = g(t).
\]

Multiply both sides by \( c \).

\[
c\phi'' + cp(t)\phi' + cq(t)\phi = cg(t)
\]

\[
(c\phi)'' + p(t)(c\phi)' + q(t)(c\phi) = cg(t)
\]

Therefore, \( y = c\phi(t) \) satisfies \( y'' + p(t)y' + q(t)y = cg(t) \), not \( y'' + p(t)y' + q(t)y = g(t) \).

This result does not contradict Theorem 3.2.2 because the theorem assumes the ODE is of the form \( y'' + p(t)y' + q(t)y = 0 \). The ODE dealt with in this problem is \( y'' + p(t)y' + q(t)y = g(t) \), a different one, so Theorem 3.2.2 does not apply.