Problem 17

If the Wronskian W of f and g is $3e^{4t}$, and if $f(t) = e^{2t}$, find $g(t)$.

Solution

The Wronskian of f and g is

$$\begin{aligned} W &= \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} \\ &= \begin{vmatrix} e^{2t} & g(t) \\ 2e^{2t} & g'(t) \end{vmatrix} \\ &= e^{2t}g'(t) - g(t)(2e^{2t}) \\ &= e^{2t}g'(t) - 2e^{2t}g(t) \end{aligned}$$

Set this result equal to $3e^{4t}$.

$$e^{2t}g'(t) - 2e^{2t}g(t) = 3e^{4t}$$

Divide both sides by e^{2t}.

$$g'(t) - 2g(t) = 3e^{2t}$$

Solve this ODE for g by multiplying both sides by an integrating factor I.

$$I = \exp\left[\int^t (-2)\, ds\right] = e^{-2t}$$

Proceed with the multiplication.

$$e^{-2t}g'(t) - 2e^{-2t}g(t) = 3$$

The left side can be written as $d/dt(Ig)$ by the chain rule.

$$\frac{d}{dt}(e^{-2t}g) = 3$$

Integrate both sides with respect to t.

$$e^{-2t}g = 3t + C$$

Therefore,

$$g(t) = 3te^{2t} + Ce^{2t}.$$