Problem 18

If the Wronskian W of f and g is t^2e^t, and if $f(t) = t$, find $g(t)$.

Solution

The Wronskian of f and g is

$$\begin{aligned} W &= \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} \\ &= \begin{vmatrix} t & g(t) \\ 1 & g'(t) \end{vmatrix} \\ &= tg'(t) - g(t)(1) \\ &= tg'(t) - g(t) \end{aligned}$$

Set this result equal to t^2e^t.

$$tg'(t) - g(t) = t^2e^t$$

Divide both sides by t.

$$g'(t) - \frac{1}{t}g(t) = te^t$$

Solve this ODE for g by multiplying both sides by an integrating factor I.

$$I = \exp\left[\int^t \left(-\frac{1}{s}\right) ds\right] = e^{-\ln t} = e^{\ln t^{-1}} = t^{-1}$$

Proceed with the multiplication.

$$\frac{1}{t}g'(t) - \frac{1}{t^2}g(t) = e^t$$

The left side can be written as $d/dt(Ig)$ by the chain rule.

$$\frac{d}{dt}\left(\frac{1}{t}g\right) = e^t$$

Integrate both sides with respect to t.

$$\frac{1}{t}g = e^t + C$$

Therefore,

$$g(t) = te^t + Ct.$$