Problem 26

In each of Problems 24 through 27, verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

$$x^2y'' - x(x+2)y' + (x+2)y = 0, \quad x > 0; \qquad y_1(x) = x, \quad y_2(x) = xe^x$$

Solution

Check that y_1 is a solution of the ODE.

$$x^2y_1'' - x(x+2)y_1' + (x+2)y_1 \stackrel{?}{=} 0$$

$$x^2\frac{d^2}{dx^2}(x) - x(x+2)\frac{d}{dx}(x) + (x+2)(x) \stackrel{?}{=} 0$$

$$x^2(0) - x(x+2)(1) + (x+2)(x) \stackrel{?}{=} 0$$

$$-x(x+2) + x(x+2) \stackrel{?}{=} 0$$

$$0 = 0$$

Now check that y_2 is a solution of the ODE.

$$x^2y_2'' - x(x+2)y_2' + (x+2)y_2 \stackrel{?}{=} 0$$

$$x^2\frac{d^2}{dx^2}(xe^x) - x(x+2)\frac{d}{dx}(xe^x) + (x+2)(xe^x) \stackrel{?}{=} 0$$

$$x^2\frac{d}{dx}(e^x + xe^x) - x(x+2)(e^x + xe^x) + x(x+2)e^x \stackrel{?}{=} 0$$

$$x^2(e^x + e^x + xe^x) - x(x+2)(e^x + xe^x) + x(x+2)e^x \stackrel{?}{=} 0$$

$$\cancel{x^2(x+2)e^x} - \cancel{x(x+2)e^x} - \cancel{x^2(x+2)e^x} + \cancel{x(x+2)e^x} \stackrel{?}{=} 0$$

$$0 = 0$$

Calculate $W(y_1, y_2)$, the Wronskian of y_1 and y_2.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} \\ &= x(e^x + xe^x) - xe^x(1) \\ &= xe^x + x^2e^x - xe^x \\ &= x^2e^x \end{aligned}$$

Since $W(y_1, y_2) \neq 0$ for $x > 0$, y_1 and y_2 form a fundamental set of solutions.