Problem 27

In each of Problems 24 through 27, verify that the functions y_1 and y_2 are solutions of the given differential equation. Do they constitute a fundamental set of solutions?

$$(1 - x\cot x)y'' - xy' + y = 0, \quad 0 < x < \pi; \qquad y_1(x) = x, \quad y_2(x) = \sin x$$

Solution

Check that y_1 is a solution of the ODE.

$$(1 - x\cot x)y_1'' - xy_1' + y_1 \overset{?}{=} 0$$

$$(1 - x\cot x)\frac{d^2}{dx^2}(x) - x\frac{d}{dx}(x) + x \overset{?}{=} 0$$

$$(1 - x\cot x)(0) - x(1) + x \overset{?}{=} 0$$

$$-x + x \overset{?}{=} 0$$

$$0 = 0$$

Now check that y_2 is a solution of the ODE.

$$(1 - x\cot x)y_2'' - xy_2' + y_2 \overset{?}{=} 0$$

$$(1 - x\cot x)\frac{d^2}{dx^2}(\sin x) - x\frac{d}{dx}(\sin x) + \sin x \overset{?}{=} 0$$

$$(1 - x\cot x)(-\sin x) - x(\cos x) + \sin x \overset{?}{=} 0$$

$$\cancel{-\sin x} + \cancel{x\cos x} - \cancel{x\cos x} + \cancel{\sin x} \overset{?}{=} 0$$

$$0 = 0$$

Calculate $W(y_1, y_2)$, the Wronskian of y_1 and y_2.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} x & \sin x \\ 1 & \cos x \end{vmatrix} \\ &= x(\cos x) - \sin x(1) \\ &= x\cos x - \sin x \end{aligned}$$

Since $W(y_1, y_2) \neq 0$ for $0 < x < \pi$, y_1 and y_2 form a fundamental set of solutions.