Problem 31

In each of Problems 29 through 32, find the Wronskian of two solutions of the given differential equation without solving the equation.

\[ x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad \text{Bessel’s equation} \]

Solution

Suppose that \( y_1 \) and \( y_2 \) are two solutions to the ODE. They then satisfy

\[
\begin{align*}
x^2y_1'' + xy_1' + (x^2 - \nu^2)y_1 &= 0 \\
x^2y_2'' + xy_2' + (x^2 - \nu^2)y_2 &= 0.
\end{align*}
\]

Multiply both sides of the first equation by \(-y_2\) and multiply both sides of the second equation by \(y_1\).

\[
\begin{align*}
x^2y_1''y_2 - xy_1y_2 - (x^2 - \nu^2)y_1y_2 &= 0 \\
x^2y_1y_2'' + xy_1y_2' + (x^2 - \nu^2)y_1y_2 &= 0
\end{align*}
\]

Add the respective sides of each equation.

\[
-x^2y_1'y_2 + x^2y_1'y_2 - xy_1y_2 + xy_1y_2' = 0
\]

Factor the left side.

\[
x^2(y_1'y_2'' - y''y_2) + x(y_1'y_2 - y'y_2) = 0
\]

Note that the Wronskian of \( y_1 \) and \( y_2 \) is

\[
W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2 \implies W'(y_1, y_2) = y_1y_2'' + y_1'y_2' - y_1''y_2 - y_1'y_2' = y_1y''_2 - y_1'y_2,
\]

so the previous equation can be written as

\[
x^2W' + xW = 0.
\]

Solve for \( W \) by separating variables.

\[
\frac{dW}{W} = -\frac{dx}{x}
\]

Integrate both sides.

\[
\ln|W| = -\ln|x| + C \\
\ln|W| + \ln|x| = C \\
\ln|W| = C
\]

Exponentiate both sides.

\[
|W| = e^C
\]

Introduce \( \pm \) on the right side to remove the absolute value sign.

\[
xW = \pm e^C
\]

Therefore, using a new constant \( A \) for \( \pm e^C \), the Wronskian of two solutions of the ODE is

\[
W(x) = \frac{A}{x}.
\]