Problem 34

If the differential equation $ty'' + 2y' + t e^t y = 0$ has $y_1$ and $y_2$ as a fundamental set of solutions and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$. 

Solution

Since $y_1$ and $y_2$ are both solutions to the ODE, they satisfy

\begin{align*}
ty_1'' + 2y_1' + t e^t y_1 &= 0 \\
ty_2'' + 2y_2' + t e^t y_2 &= 0.
\end{align*}

Multiply both sides of the first equation by $-y_2$ and multiply both sides of the second equation by $y_1$.

\begin{align*}
-t y_1'' y_2 - 2 y_1' y_2' - t e^t y_1 y_2 &= 0 \\
ty_1 y_2'' + 2 y_1' y_2' + t e^t y_1 y_2 &= 0
\end{align*}

Add the respective sides of each equation.

\begin{align*}
-t y_1'' y_2 + t y_1 y_2'' - 2 y_1' y_2 + 2 y_1 y_2' &= 0 \\
W'(y_1, y_2) &= y_1' y_2'' - y_1'' y_2' - y_1 y_2' - y_1' y_2 = y_1 y_2'' - y_1'' y_2.
\end{align*}

so the previous equation can be written as

\begin{align*}
tW' + 2W &= 0.
\end{align*}

Solve this ODE by separating variables.

\begin{align*}
\frac{dW}{W} &= -\frac{2}{t} dt \\
\ln |W| &= -2 \ln |t| + C \\
\ln |W| + 2 \ln |t| &= C \\
\ln |t^2 W| &= C
\end{align*}

Exponentiate both sides.

\begin{align*}
|t^2 W| &= e^C \\
\int t^2 W &= \pm e^C
\end{align*}

Introduce $\pm$ on the right side to remove the absolute value sign.
Using a new constant $A$ for $\pm e^C$, the Wronskian of two solutions of the ODE is

$$W(t) = \frac{A}{t^2}.$$  

Use the fact that $W = 2$ when $t = 1$ to determine $A$.

$$W(1) = A = 2$$

The Wronskian is then

$$W(t) = \frac{2}{t^2}.$$  

Therefore,

$$W(5) = \frac{2}{25}.$$