Problem 34

If the differential equation $ty'' + 2y' + te^t y = 0$ has y_1 and y_2 as a fundamental set of solutions and if $W(y_1, y_2)(1) = 2$, find the value of $W(y_1, y_2)(5)$.

Solution

Since y_1 and y_2 are both solutions to the ODE, they satisfy

$$\begin{aligned} ty_1'' + 2y_1' + te^t y_1 &= 0 \\ ty_2'' + 2y_2' + te^t y_2 &= 0. \end{aligned}$$

Multiply both sides of the first equation by $-y_2$ and multiply both sides of the second equation by y_1.

$$\begin{aligned} -ty_1''y_2 - 2y_1'y_2 - te^t y_1 y_2 &= 0 \\ ty_1 y_2'' + 2y_1 y_2' + te^t y_1 y_2 &= 0 \end{aligned}$$

Add the respective sides of each equation.

$$-ty_1''y_2 + ty_1 y_2'' - 2y_1'y_2 + 2y_1 y_2' = 0$$

Factor the left side.

$$t(y_1 y_2'' - y_1'' y_2) + 2(y_1 y_2' - y_1' y_2) = 0$$

Note that the Wronskian of y_1 and y_2 is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2 \quad \Rightarrow \quad W'(y_1, y_2) = \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'} = y_1 y_2'' - y_1'' y_2,$$

so the previous equation can be written as

$$tW' + 2W = 0.$$

Solve this ODE by separating variables.

$$\frac{dW}{W} = -\frac{2}{t}\,dt$$

Integrate both sides.

$$\begin{aligned} \ln|W| &= -2\ln|t| + C \\ \ln|W| + 2\ln|t| &= C \\ \ln|W| + \ln|t^2| &= C \\ \ln|t^2 W| &= C \end{aligned}$$

Exponentiate both sides.

$$|t^2 W| = e^C$$

Introduce $\pm$ on the right side to remove the absolute value sign.

$$t^2 W = \pm e^C$$

Using a new constant A for $\pm e^C$, the Wronskian of two solutions of the ODE is

$$W(t) = \frac{A}{t^2}.$$

Use the fact that $W = 2$ when $t = 1$ to determine A.

$$W(1) = A = 2$$

The Wronskian is then

$$W(t) = \frac{2}{t^2}.$$

Therefore,

$$W(5) = \frac{2}{25}.$$