Problem 42

In each of Problems 42 through 45, use the result of Problem 41 to determine whether the given equation is exact. If it is, then solve the equation.

\[ y'' + xy' + y = 0 \]

Solution

Notice that the last two terms can be written as \((xy)\)'.

\[ y'' + (xy)' = 0 \]

Integrate both sides with respect to \(x\).

\[ y' + xy = C_1 \]

To solve this first-order linear inhomogeneous ODE, multiply both sides by an integrating factor \(I\).

\[ I = \exp\left(\int^x s \, ds\right) = e^{x^2/2} \]

Proceed with the multiplication.

\[ e^{x^2/2}y' + xe^{x^2/2}y = C_1e^{x^2/2} \]

The left side can be written as \((e^{x^2/2}y)'\) by the product rule.

\[ (e^{x^2/2}y)' = C_1e^{x^2/2} \]

Integrate both sides with respect to \(x\) once more.

\[ e^{x^2/2}y = \int^x C_1e^{s^2/2} \, ds + C_2 \]

Therefore,

\[ y(x) = e^{-x^2/2} \int^x C_1e^{s^2/2} \, ds + C_2e^{-x^2/2} = C_1 \int^x e^{(s^2-x^2)/2} \, ds + C_2e^{-x^2/2}. \]