Problem 46

The Adjoint Equation. If a second order linear homogeneous equation is not exact, it can be made exact by multiplying by an appropriate integrating factor $\mu(x)$. Thus we require that $\mu(x)$ be such that

$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0$$

can be written in the form

$$[\mu(x)P(x)y']' + [f(x)y]' = 0.$$

By equating coefficients in these two equations and eliminating $f(x)$, show that the function μ must satisfy

$$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0.$$

This equation is known as the adjoint of the original equation and is important in the advanced theory of differential equations. In general, the problem of solving the adjoint differential equation is as difficult as that of solving the original equation, so only occasionally is it possible to find an integrating factor for a second order equation.

Solution

Suppose we have the second-order ODE,

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

To make it exact, multiply both sides by an integrating factor $\mu = \mu(x)$.

$$\mu(x)P(x)y'' + \mu(x)Q(x)y' + \mu(x)R(x)y = 0 \tag{1}$$

Now that it's exact, it can be written in the form,

$$[\mu(x)P(x)y']' + [f(x)y]' = 0.$$

Expand the left side.

$$\mu'(x)P(x)y' + \mu(x)P'(x)y' + \mu(x)P(x)y'' + f'(x)y + f(x)y' = 0$$

Factor it now.

$$\mu(x)P(x)y'' + [\mu'(x)P(x) + \mu(x)P'(x) + f(x)]y' + f'(x)y = 0$$

Equate the coefficients with those of equation (1).

$$\begin{aligned} \mu'(x)P(x) + \mu(x)P'(x) + f(x) &= \mu(x)Q(x) \\ f'(x) &= \mu(x)R(x) \end{aligned}$$

Differentiate both sides of the first equation with respect to x.

$$\mu''(x)P(x) + \mu'(x)P'(x) + \mu'(x)P'(x) + \mu(x)P''(x) + f'(x) = \mu'(x)Q(x) + \mu(x)Q'(x)$$

Substitute $\mu(x)R(x)$ for $f'(x)$.

$$\mu''(x)P(x) + \mu'(x)P'(x) + \mu'(x)P'(x) + \mu(x)P''(x) + \mu(x)R(x) = \mu'(x)Q(x) + \mu(x)Q'(x)$$

Therefore, the adjoint is

$$P\mu'' + (2P' - Q)\mu' + (P'' - Q' + R)\mu = 0.$$