Problem 47

In each of Problems 47 through 49, use the result of Problem 46 to find the adjoint of the given differential equation.

\[ x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad \text{Bessel's equation} \]

Solution

To make the ODE exact, multiply both sides by an integrating factor \( \mu = \mu(x) \).

\[ x^2\mu(x)y'' + x\mu(x)y' + (x^2 - \nu^2)\mu(x)y = 0 \]  \hspace{1cm} (1)

Now that it’s exact, it can be written in the form,

\[ [x^2\mu(x)y']' + [f(x)y]' = 0. \]

Expand the left side.

\[ 2x\mu(x)y' + x^2\mu'(x)y' + x^2\mu(x)y'' + f'(x)y + f(x)y' = 0 \]

Factor it now.

\[ x^2\mu(x)y'' + [x^2\mu(x) + 2x\mu(x) + f(x)]y' + f'(x)y = 0 \]

Equate the coefficients with those of equation (1).

\[ x^2\mu'(x) + 2x\mu(x) + f(x) = x\mu(x) \]
\[ f'(x) = (x^2 - \nu^2)\mu(x) \]

Differentiate both sides of the first equation with respect to \( x \).

\[ 2x\mu'(x) + x^2\mu''(x) + 2\mu(x) + 2x\mu'(x) + f'(x) = \mu(x) + x\mu'(x) \]

Substitute \( (x^2 - \nu^2)\mu(x) \) for \( f'(x) \).

\[ 2x\mu'(x) + x^2\mu''(x) + 2\mu(x) + 2x\mu'(x) + (x^2 - \nu^2)\mu(x) = \mu(x) + x\mu'(x) \]

Bring all terms to the left side and combine like terms.

\[ x^2\mu''(x) + 3x\mu'(x) + (x^2 - \nu^2 + 1)\mu(x) = 0 \]