Problem 47

In each of Problems 47 through 49, use the result of Problem 46 to find the adjoint of the given differential equation.

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad \text{Bessel's equation}$$

Solution

To make the ODE exact, multiply both sides by an integrating factor $\mu = \mu(x)$.

$$x^2\mu(x)y'' + x\mu(x)y' + (x^2 - \nu^2)\mu(x)y = 0 \tag{1}$$

Now that it's exact, it can be written in the form,

$$[x^2\mu(x)y']' + [f(x)y]' = 0.$$

Expand the left side.

$$2x\mu(x)y' + x^2\mu'(x)y' + x^2\mu(x)y'' + f'(x)y + f(x)y' = 0$$

Factor it now.

$$x^2\mu(x)y'' + [x^2\mu'(x) + 2x\mu(x) + f(x)]y' + f'(x)y = 0$$

Equate the coefficients with those of equation (1).

$$\begin{aligned} x^2\mu'(x) + 2x\mu(x) + f(x) &= x\mu(x) \\ f'(x) &= (x^2 - \nu^2)\mu(x) \end{aligned}$$

Differentiate both sides of the first equation with respect to x.

$$2x\mu'(x) + x^2\mu''(x) + 2\mu(x) + 2x\mu'(x) + f'(x) = \mu(x) + x\mu'(x)$$

Substitute $(x^2 - \nu^2)\mu(x)$ for $f'(x)$.

$$2x\mu'(x) + x^2\mu''(x) + 2\mu(x) + 2x\mu'(x) + (x^2 - \nu^2)\mu(x) = \mu(x) + x\mu'(x)$$

Bring all terms to the left side and combine like terms.

$$x^2\mu''(x) + 3x\mu'(x) + (x^2 - \nu^2 + 1)\mu(x) = 0$$