Problem 48

In each of Problems 47 through 49, use the result of Problem 46 to find the adjoint of the given differential equation.

\[(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \quad \text{Legendre's equation}\]

Solution

To make the ODE exact, multiply both sides by an integrating factor \(\mu = \mu(x)\).

\[(1 - x^2)\mu(x)y'' - 2x\mu(x)y' + \alpha(\alpha + 1)\mu(x)y = 0 \quad (1)\]

Now that it’s exact, it can be written in the form,

\[[(1 - x^2)\mu(x)y']' + [f(x)y]' = 0.\]

Expand the left side.

\[-2x\mu(x)y' + (1 - x^2)\mu'(x)y' + (1 - x^2)\mu(x)y'' + f'(x)y + f(x)y' = 0\]

Factor it now.

\[(1 - x^2)\mu(x)y'' + [(1 - x^2)\mu'(x) - 2x\mu(x) + f(x)]y' + f'(x)y = 0\]

Equate the coefficients with those of equation (1).

\[(1 - x^2)\mu'(x) - 2x\mu(x) + f(x) = -2x\mu(x)\]

\[f'(x) = \alpha(\alpha + 1)\mu(x)\]

Add \(2x\mu(x)\) to both sides of the first equation.

\[(1 - x^2)\mu'(x) + f(x) = 0\]

Differentiate both sides with respect to \(x\).

\[-2x\mu'(x) + (1 - x^2)\mu''(x) + f'(x) = 0\]

Substitute \(\alpha(\alpha + 1)\mu(x)\) for \(f'(x)\).

\[-2x\mu'(x) + (1 - x^2)\mu''(x) + \alpha(\alpha + 1)\mu(x) = 0\]

Therefore,

\[(1 - x^2)\mu''(x) - 2x\mu'(x) + \alpha(\alpha + 1)\mu(x) = 0.\]