Problem 49

In each of Problems 47 through 49, use the result of Problem 46 to find the adjoint of the given differential equation.

\[ y'' - xy = 0, \quad \text{Airy’s equation} \]

Solution

To make the ODE exact, multiply both sides by an integrating factor \( \mu = \mu(x) \).

\[ \mu(x)y'' - x\mu(x)y = 0 \quad (1) \]

Now that it’s exact, it can be written in the form,

\[ [\mu(x)y']' + [f(x)y]' = 0. \]

Expand the left side.

\[ \mu'(x)y' + \mu(x)y'' + f'(x)y + f(x)y' = 0 \]

Factor it now.

\[ \mu(x)y'' + [\mu'(x) + f(x)]y' + f'(x)y = 0 \]

Equate the coefficients with those of equation (1).

\[ \mu'(x) + f(x) = 0 \]
\[ f'(x) = -x\mu(x) \]

Differentiate both sides of the first equation with respect to \( x \).

\[ \mu''(x) + f'(x) = 0 \]

Substitute \(-x\mu(x)\) for \( f'(x) \).

\[ \mu''(x) - x\mu(x) = 0 \]