Problem 11

In each of Problems 7 through 16, find the general solution of the given differential equation.

$$y'' + 6y' + 13y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + 6(re^{rt}) + 13(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 6r + 13 = 0$$

$$r = \frac{-6 \pm \sqrt{36 - 4(13)(1)}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i$$

$$r = \{-3 - 2i, -3 + 2i\}$$

Two solutions to the ODE are $y = e^{(-3-2i)t}$ and $y = e^{(-3+2i)t}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_1e^{(-3-2i)t} + C_2e^{(-3+2i)t} \\
&= C_1e^{-3t-2it} + C_2e^{-3t+2it} \\
&= C_1e^{-3t}e^{-2it} + C_2e^{-3t}e^{2it} \\
&= C_1e^{-3t}[\cos(-2t) + i\sin(-2t)] + C_2e^{-3t}(\cos 2t + i\sin 2t) \\
&= C_1e^{-3t}(\cos 2t - i\sin 2t) + C_2e^{-3t}(\cos 2t + i\sin 2t) \\
&= C_1e^{-3t}\cos 2t - iC_1e^{-3t}\sin 2t + C_2e^{-3t}\cos 2t + iC_2e^{-3t}\sin 2t \\
&= (C_1 + C_2)e^{-3t}\cos 2t + (-iC_1 + iC_2)e^{-3t}\sin 2t
\end{aligned}$$

Therefore, using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(t) = C_3e^{-3t}\cos 2t + C_4e^{-3t}\sin 2t.$$