Problem 15

In each of Problems 7 through 16, find the general solution of the given differential equation.

\[ y'' + y' + 1.25y = 0 \]

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form \( y = e^{rt} \).

\[ y = e^{rt} \rightarrow y' = re^{rt} \rightarrow y'' = r^2e^{rt} \]

Substitute these expressions into the ODE.

\[ r^2e^{rt} + re^{rt} + 1.25(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 + r + 1.25 = 0 \]

\[ r = \frac{-1 \pm \sqrt{1 - 4(1.25)(1)}}{2} = \frac{-1 \pm \sqrt{-4}}{2} = \frac{-1 \pm 2i}{2} = \frac{-1}{2} \pm i \]

\[ r = \left\{ \frac{-1}{2} - i, \frac{-1}{2} + i \right\} \]

Two solutions to the ODE are \( y = e^{(-1/2-i)t} \) and \( y = e^{(-1/2+i)t} \), so the general solution is a linear combination of the two.

\[ y(t) = C_1e^{(-1/2-i)t} + C_2e^{(-1/2+i)t} \]

\[ = C_1e^{-t/2}e^{-it} + C_2e^{-t/2}e^{it} \]

\[ = C_1e^{-t/2}[\cos(-t) + i\sin(-t)] + C_2e^{-t/2}(\cos t + i\sin t) \]

\[ = C_1e^{-t/2}(\cos t - i\sin t) + C_2e^{-t/2}(\cos t + i\sin t) \]

\[ = C_1e^{-t/2} \cos t - iC_1e^{-t/2} \sin t + C_2e^{-t/2} \cos t + iC_2e^{-t/2} \sin t \]

\[ = (C_1 + C_2)e^{-t/2} \cos t + (-iC_1 + iC_2)e^{-t/2} \sin t \]

Therefore, using \( C_3 \) for \( C_1 + C_2 \) and \( C_4 \) for \(-iC_1 + iC_2\), the real general solution is

\[ y(t) = C_3e^{-t/2} \cos t + C_4e^{-t/2} \sin t. \]