Problem 20

In each of Problems 17 through 22, find the solution of the given initial value problem. Sketch the graph of the solution and describe its behavior for increasing \( t \).

\[
y'' + y = 0, \quad y(\pi/3) = 2, \quad y'(\pi/3) = -4
\]

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form \( y = e^{rt} \).

\[
y = e^{rt} \rightarrow y' = re^{rt} \rightarrow y'' = r^2e^{rt}
\]

Substitute these expressions into the ODE.

\[
r^2e^{rt} + e^{rt} = 0
\]

Divide both sides by \( e^{rt} \).

\[
r^2 + 1 = 0
\]

\[
r = \{-i, i\}
\]

Two solutions to the ODE are \( y = e^{-it} \) and \( y = e^{it} \), so the general solution is a linear combination of the two.

\[
y(t) = C_1e^{-it} + C_2e^{it}
\]

\[
= C_1[\cos(t) + i \sin(t)] + C_2[\cos(t) + i \sin(t)]
\]

\[
= C_1 \cos t - iC_1 \sin t + C_2 \cos t + iC_2 \sin t
\]

\[
= (C_1 + C_2) \cos t + (iC_1 + iC_2) \sin t
\]

Using \( C_3 \) for \( C_1 + C_2 \) and \( C_4 \) for \( -iC_1 + iC_2 \), the real general solution is

\[
y(t) = C_3 \cos t + C_4 \sin t.
\]

Take a derivative of it.

\[
y'(t) = -C_3 \sin t + C_4 \cos t
\]

Apply the initial conditions now to determine \( C_3 \) and \( C_4 \).

\[
y\left(\frac{\pi}{3}\right) = C_3 \left(\frac{1}{2}\right) + C_4 \left(\frac{\sqrt{3}}{2}\right) = 2
\]

\[
y'\left(\frac{\pi}{3}\right) = -C_3 \left(\frac{\sqrt{3}}{2}\right) + C_4 \left(\frac{1}{2}\right) = -4
\]

Solving this system of equations yields \( C_3 = 1 + 2\sqrt{3} \) and \( C_4 = -2 + \sqrt{3} \). Therefore,

\[
y(t) = (1 + 2\sqrt{3}) \cos t + (-2 + \sqrt{3}) \sin t.
\]
The solution oscillates with a constant amplitude and period.

\[ y(t) = (1 + 2\sqrt{3})\cos t + (-2 + \sqrt{3})\sin t \]