Problem 23

Consider the initial value problem

\[ 3u'' - u' + 2u = 0, \quad u(0) = 2, \quad u'(0) = 0. \]

(a) Find the solution \( u(t) \) of this problem.

(b) For \( t > 0 \), find the first time at which \( |u(t)| = 10 \).

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form \( u = e^{rt} \).

\[ u = e^{rt} \quad \Rightarrow \quad u' = re^{rt} \quad \Rightarrow \quad u'' = r^2e^{rt} \]

Substitute these expressions into the ODE.

\[ 3(r^2e^{rt}) - (re^{rt}) + 2(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ 3r^2 - r + 2 = 0 \]

\[ r = \frac{1 \pm \sqrt{1 - 4(3)(2)}}{2(3)} = \frac{1 \pm \sqrt{23}}{6} = \frac{1}{6} \pm \frac{i\sqrt{23}}{6} \]

\[ r = \begin{cases} \frac{1}{6} - i\frac{\sqrt{23}}{6}, & \frac{1}{6} + i\frac{\sqrt{23}}{6} \end{cases} \]

Two solutions to the ODE are \( u = e^{(1/6 - i\sqrt{23}/6)t} \) and \( u = e^{(1/6 + i\sqrt{23}/6)t} \), so the general solution is a linear combination of the two.

\[ u(t) = C_1e^{(1/6 - i\sqrt{23}/6)t} + C_2e^{(1/6 + i\sqrt{23}/6)t} \]

\[ = C_1e^{t/6 - it\sqrt{23}/6} + C_2e^{t/6 + it\sqrt{23}/6} \]

\[ = C_1e^{t/6}e^{-it\sqrt{23}/6} + C_2e^{t/6}e^{it\sqrt{23}/6} \]

\[ = C_1e^{t/6} \left[ \cos \left( -\frac{\sqrt{23}}{6}t \right) + i \sin \left( -\frac{\sqrt{23}}{6}t \right) \right] + C_2e^{t/6} \left[ \cos \left( \frac{\sqrt{23}}{6}t \right) + i \sin \left( \frac{\sqrt{23}}{6}t \right) \right] \]

\[ = C_1e^{t/6} \left[ \cos \left( \frac{\sqrt{23}}{6}t \right) - i \sin \left( \frac{\sqrt{23}}{6}t \right) \right] + C_2e^{t/6} \left[ \cos \left( \frac{\sqrt{23}}{6}t \right) + i \sin \left( \frac{\sqrt{23}}{6}t \right) \right] \]

\[ = C_1e^{t/6} \cos \left( \frac{\sqrt{23}}{6}t \right) - iC_1e^{t/6} \sin \left( \frac{\sqrt{23}}{6}t \right) + C_2e^{t/6} \cos \left( \frac{\sqrt{23}}{6}t \right) + iC_2e^{t/6} \sin \left( \frac{\sqrt{23}}{6}t \right) \]

\[ = (C_1 + C_2)e^{t/6} \cos \left( \frac{\sqrt{23}}{6}t \right) + (-iC_1 + iC_2)e^{t/6} \sin \left( \frac{\sqrt{23}}{6}t \right) \]

Using \( C_3 \) for \( C_1 + C_2 \) and \( C_4 \) for \(-iC_1 + iC_2\), the real general solution is

\[ u(t) = C_3e^{t/6} \cos \left( \frac{\sqrt{23}}{6}t \right) + C_4e^{t/6} \sin \left( \frac{\sqrt{23}}{6}t \right). \]

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Take a derivative of it.

\[
  u'(t) = \frac{C_3}{6} e^{t/6} \cos \left( \frac{\sqrt{23}}{6} t \right) - C_3 \frac{\sqrt{23}}{6} e^{t/6} \sin \left( \frac{\sqrt{23}}{6} t \right) \\
  + \frac{C_4}{6} e^{t/6} \sin \left( \frac{\sqrt{23}}{6} t \right) + C_4 \frac{\sqrt{23}}{6} e^{t/6} \cos \left( \frac{\sqrt{23}}{6} t \right)
\]

Apply the initial conditions now to determine \( C_3 \) and \( C_4 \).

\[
  u(0) = C_3 = 2 \\
  u'(0) = \frac{C_3}{6} + C_4 \frac{\sqrt{23}}{6} = 0
\]

Solving this system of equations yields \( C_3 = 2 \) and \( C_4 = -\frac{2}{\sqrt{23}} \). Therefore,

\[
  u(t) = 2 e^{t/6} \cos \left( \frac{\sqrt{23}}{6} t \right) - \frac{2}{\sqrt{23}} e^{t/6} \sin \left( \frac{\sqrt{23}}{6} t \right).
\]

Based on the graph, the smallest time for which the amplitude is 10 is \( t \approx 10.8 \).