Problem 23

Consider the initial value problem

$$3u'' - u' + 2u = 0, \qquad u(0) = 2, \quad u'(0) = 0.$$

(a) Find the solution $u(t)$ of this problem.

(b) For $t > 0$, find the first time at which $|u(t)| = 10$.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $u = e^{rt}$.

$$u = e^{rt} \quad \rightarrow \quad u' = re^{rt} \quad \rightarrow \quad u'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$3(r^2 e^{rt}) - (re^{rt}) + 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$3r^2 - r + 2 = 0$$

$$r = \frac{1 \pm \sqrt{1 - 4(3)(2)}}{2(3)} = \frac{1 \pm \sqrt{-23}}{6} = \frac{1 \pm i\sqrt{23}}{6} = \frac{1}{6} \pm i\frac{\sqrt{23}}{6}$$

$$r = \left\{\frac{1}{6} - i\frac{\sqrt{23}}{6}, \frac{1}{6} + i\frac{\sqrt{23}}{6}\right\}$$

Two solutions to the ODE are $u = e^{(1/6 - i\sqrt{23}/6)t}$ and $u = e^{(1/6 + i\sqrt{23}/6)t}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
u(t) &= C_1 e^{(1/6 - i\sqrt{23}/6)t} + C_2 e^{(1/6 + i\sqrt{23}/6)t} \\
&= C_1 e^{t/6 - it\sqrt{23}/6} + C_2 e^{t/6 + it\sqrt{23}/6} \\
&= C_1 e^{t/6} e^{-it\sqrt{23}/6} + C_2 e^{t/6} e^{it\sqrt{23}/6} \\
&= C_1 e^{t/6}\left[\cos\left(-\frac{\sqrt{23}}{6}t\right) + i\sin\left(-\frac{\sqrt{23}}{6}t\right)\right] + C_2 e^{t/6}\left[\cos\left(\frac{\sqrt{23}}{6}t\right) + i\sin\left(\frac{\sqrt{23}}{6}t\right)\right] \\
&= C_1 e^{t/6}\left[\cos\left(\frac{\sqrt{23}}{6}t\right) - i\sin\left(\frac{\sqrt{23}}{6}t\right)\right] + C_2 e^{t/6}\left[\cos\left(\frac{\sqrt{23}}{6}t\right) + i\sin\left(\frac{\sqrt{23}}{6}t\right)\right] \\
&= C_1 e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) - iC_1 e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right) + C_2 e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) + iC_2 e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right) \\
&= (C_1 + C_2)e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) + (-iC_1 + iC_2)e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right)
\end{aligned}$$

Using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$u(t) = C_3 e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) + C_4 e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right).$$

Take a derivative of it.

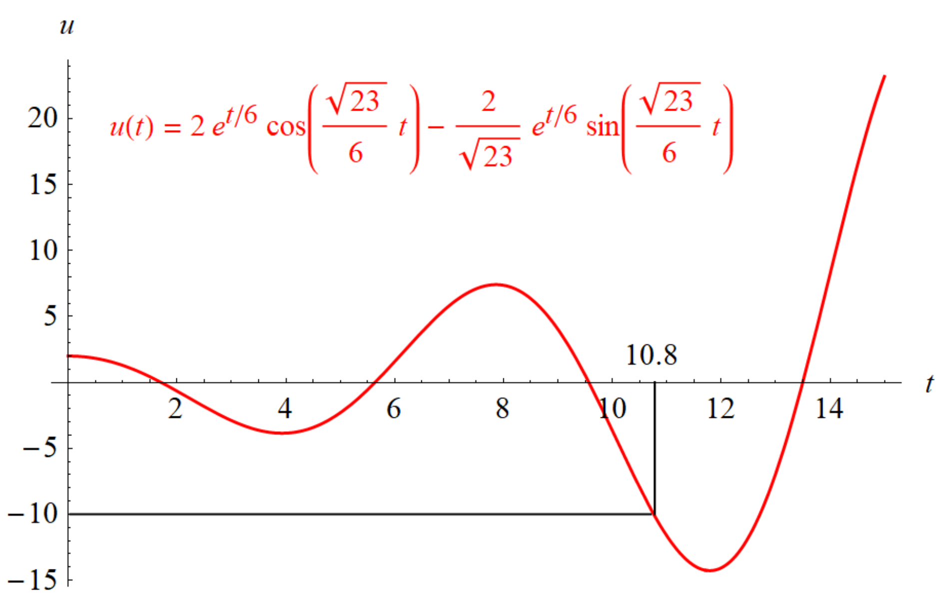
$$u'(t) = \frac{C_3}{6}e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) - C_3\frac{\sqrt{23}}{6}e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right) + \frac{C_4}{6}e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right) + C_4\frac{\sqrt{23}}{6}e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right)$$

Apply the initial conditions now to determine C_3 and C_4.

$$u(0) = C_3 = 2$$
$$u'(0) = \frac{C_3}{6} + C_4\frac{\sqrt{23}}{6} = 0$$

Solving this system of equations yields $C_3 = 2$ and $C_4 = -2/\sqrt{23}$. Therefore,

$$u(t) = 2e^{t/6}\cos\left(\frac{\sqrt{23}}{6}t\right) - \frac{2}{\sqrt{23}}e^{t/6}\sin\left(\frac{\sqrt{23}}{6}t\right).$$



Based on the graph, the smallest time for which the amplitude is 10 is $t \approx 10.8$.