Problem 25

Consider the initial value problem

$$y'' + 2y' + 6y = 0, \qquad y(0) = 2, \quad y'(0) = \alpha \geq 0.$$

(a) Find the solution $y(t)$ of this problem.

(b) Find α such that $y = 0$ when $t = 1$.

(c) Find, as a function of α, the smallest positive value of t for which $y = 0$.

(d) Determine the limit of the expression found in part (c) as $\alpha \to \infty$.

Solution

Part (a)

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \to \quad y' = re^{rt} \quad \to \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 2(re^{rt}) + 6(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2r + 6 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(6)}}{2(1)} = \frac{-2 \pm \sqrt{-20}}{2} = \frac{-2 \pm 2i\sqrt{5}}{2} = -1 \pm i\sqrt{5}$$

$$r = \{-1 - i\sqrt{5}, -1 + i\sqrt{5}\}$$

Two solutions to the ODE are $y = e^{(-1-i\sqrt{5})t}$ and $y = e^{(-1+i\sqrt{5})t}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_1 e^{(-1-i\sqrt{5})t} + C_2 e^{(-1+i\sqrt{5})t} \\
&= C_1 e^{-t-i\sqrt{5}t} + C_2 e^{-t+i\sqrt{5}t} \\
&= C_1 e^{-t} e^{-i\sqrt{5}t} + C_2 e^{-t} e^{i\sqrt{5}t} \\
&= C_1 e^{-t}[\cos(-\sqrt{5}t) + i\sin(-\sqrt{5}t)] + C_2 e^{-t}[\cos(\sqrt{5}t) + i\sin(\sqrt{5}t)] \\
&= C_1 e^{-t}[\cos(\sqrt{5}t) - i\sin(\sqrt{5}t)] + C_2 e^{-t}[\cos(\sqrt{5}t) + i\sin(\sqrt{5}t)] \\
&= C_1 e^{-t}\cos(\sqrt{5}t) - iC_1 e^{-t}\sin(\sqrt{5}t) + C_2 e^{-t}\cos(\sqrt{5}t) + iC_2 e^{-t}\sin(\sqrt{5}t) \\
&= (C_1 + C_2)e^{-t}\cos(\sqrt{5}t) + (-iC_1 + iC_2)e^{-t}\sin(\sqrt{5}t)
\end{aligned}$$

Using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(t) = C_3 e^{-t}\cos(\sqrt{5}t) + C_4 e^{-t}\sin(\sqrt{5}t).$$

Take a derivative of it.

$$y'(t) = -C_3 e^{-t}\cos(\sqrt{5}t) - \sqrt{5}C_3 e^{-t}\sin(\sqrt{5}t) - C_4 e^{-t}\sin(\sqrt{5}t) + \sqrt{5}C_4 e^{-t}\cos(\sqrt{5}t)$$

Apply the initial conditions now to determine C_3 and C_4.

$$\begin{aligned} y(0) &= C_3 = 2 \\ y'(0) &= -C_3 + \sqrt{5}C_4 = \alpha \end{aligned}$$

Solving this system of equations yields $C_3 = 2$ and $C_4 = (\alpha + 2)/\sqrt{5}$. Therefore,

$$y(t) = 2e^{-t}\cos(\sqrt{5}t) + \frac{\alpha+2}{\sqrt{5}}e^{-t}\sin(\sqrt{5}t).$$

Part (b)

Set $y = 0$ and $t = 1$ in the result of part (a) and solve for α.

$$\begin{aligned} 0 &= 2e^{-1}\cos\sqrt{5} + \frac{\alpha+2}{\sqrt{5}}e^{-1}\sin\sqrt{5} \\ \frac{\alpha+2}{\sqrt{5}}e^{-1}\sin\sqrt{5} &= -2e^{-1}\cos\sqrt{5} \\ (\alpha+2)\sin\sqrt{5} &= -2\sqrt{5}\cos\sqrt{5} \\ \alpha + 2 &= -2\sqrt{5}\frac{\cos\sqrt{5}}{\sin\sqrt{5}} \end{aligned}$$

$$\begin{aligned} \alpha &= -2 - 2\sqrt{5}\frac{\cos\sqrt{5}}{\sin\sqrt{5}} \\ &= -2\left(1 + \sqrt{5}\frac{\cos\sqrt{5}}{\sin\sqrt{5}}\right) \\ &= -2(1 + \sqrt{5}\cot\sqrt{5}) \\ &\approx 1.51. \end{aligned}$$

Part (c)

Set $y = 0$ in the result of part (a) and solve for t.

$$\begin{aligned} 0 &= 2e^{-t}\cos(\sqrt{5}t) + \frac{\alpha+2}{\sqrt{5}}e^{-t}\sin(\sqrt{5}t) \\ 0 &= 2\cos(\sqrt{5}t) + \frac{\alpha+2}{\sqrt{5}}\sin(\sqrt{5}t) \\ \frac{\alpha+2}{\sqrt{5}}\sin(\sqrt{5}t) &= -2\cos(\sqrt{5}t) \\ \tan(\sqrt{5}t) &= -\frac{2\sqrt{5}}{\alpha+2} \\ \sqrt{5}t &= \tan^{-1}\left(-\frac{2\sqrt{5}}{\alpha+2}\right) + n\pi \end{aligned}$$

To obtain the smallest positive value of t, we choose $n = 1$.

$$\sqrt{5}t = \tan^{-1}\left(-\frac{2\sqrt{5}}{\alpha+2}\right) + \pi$$

Therefore,

$$t_{\min} = \frac{1}{\sqrt{5}}\left(-\tan^{-1}\frac{2\sqrt{5}}{\alpha+2} + \pi\right).$$

Part (d)

Take the limit of $t_{\min}$ as $\alpha \to \infty$.

$$\begin{aligned}\lim_{\alpha\to\infty} t_{\min} &= \lim_{\alpha\to\infty}\frac{1}{\sqrt{5}}\left(-\tan^{-1}\frac{2\sqrt{5}}{\alpha+2} + \pi\right)\\ &= \frac{1}{\sqrt{5}}(-\tan^{-1}0 + \pi)\\ &= \frac{\pi}{\sqrt{5}}\\ &\approx 1.40\end{aligned}$$