Problem 26

Consider the initial value problem

\[ y'' + 2ay' + (a^2 + 1)y = 0, \quad y(0) = 1, \quad y'(0) = 0. \]

(a) Find the solution \( y(t) \) of this problem.

(b) For \( a = 1 \) find the smallest \( T \) such that \( |y(t)| < 0.1 \) for \( t > T \).

(c) Repeat part (b) for \( a = 1/4, 1/2, \) and \( 2 \).

(d) Using the results of parts (b) and (c), plot \( T \) versus \( a \) and describe the relation between \( T \) and \( a \).

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form

\[ y(t) = e^{rt}. \]

Substitute these expressions into the ODE.

\[ r^2 e^{rt} + 2a(re^{rt}) + (a^2 + 1)(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ r^2 + 2ar + (a^2 + 1) = 0 \]

\[ r = \frac{-2a \pm \sqrt{4a^2 - 4(1)(a^2 + 1)}}{2} = \frac{-2a \pm \sqrt{-4}}{2} = \frac{-2a \pm 2i}{2} = -a \pm i \]

Two solutions to the ODE are \( y = e^{(-a-i)t} \) and \( y = e^{(-a+i)t} \), so the general solution is a linear combination of the two.

\[ y(t) = C_1 e^{(-a-i)t} + C_2 e^{(-a+i)t} \]

Using \( C_3 \) for \( C_1 + C_2 \) and \( C_4 \) for \(-iC_1 + iC_2\), the real general solution is

\[ y(t) = C_3 e^{-at} \cos t + C_4 e^{-at} \sin t. \]

Take a derivative of it.

\[ y'(t) = -C_3ae^{-at} \cos t - C_3e^{-at} \sin t - C_4ae^{-at} \sin t + C_4e^{-at} \cos t \]

www.stemjock.com
Apply the initial conditions now to determine \( C_3 \) and \( C_4 \).

\[
\begin{align*}
y(0) &= C_3 = 1 \\
y'(0) &= -C_3a + C_4 = 0
\end{align*}
\]

Solving this system of equations yields \( C_3 = 1 \) and \( C_4 = a \). Therefore,

\[
y(t) = e^{-at} \cos t + ae^{-at} \sin t.
\]

Based on the graph for \( a = 1/4 \), the smallest \( T \) such that \( |y(t)| < 0.1 \) for \( t > T \) is about 7.43.

Based on the graph for \( a = 1/2 \), the smallest \( T \) such that \( |y(t)| < 0.1 \) for \( t > T \) is about 4.30.
Based on the graph for \( a = 1 \), the smallest \( T \) such that \(|y(t)| < 0.1\) for \( t > T \) is about 1.88.

\[
y(t) = e^{-t} \cos t + e^{-t} \sin t
\]

Based on the graph for \( a = 2 \), the smallest \( T \) such that \(|y(t)| < 0.1\) for \( t > T \) is about 1.51.

\[
y(t) = e^{-2t} \cos t + 2 e^{-2t} \sin t
\]
Plotting the points, \((0.25, 7.43)\), \((0.5, 4.30)\), \((1, 1.88)\), and \((2, 1.51)\), we obtain the following graph.

\begin{center}
\begin{tikzpicture}
\begin{axis}[
    xlabel={$a$},
    ylabel={$T$},
    xmin=0, xmax=2.5,
    ymin=0, ymax=7,
    xtick={0,0.5,1,1.5,2},
    ytick={1,2,3,4,5,6,7},
    grid=both
]
\addplot[only marks] coordinates {
    (0.25, 7.43)
    (0.5, 4.30)
    (1, 1.88)
    (2, 1.51)
};
\end{axis}
\end{tikzpicture}
\end{center}

It seems that \(T\) and \(a\) are inversely related.