Problem 26

Consider the initial value problem

$$y'' + 2ay' + (a^2 + 1)y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

(a) Find the solution $y(t)$ of this problem.

(b) For $a = 1$ find the smallest T such that $|y(t)| < 0.1$ for $t > T$.

(c) Repeat part (b) for $a = 1/4$, $1/2$, and 2.

(d) Using the results of parts (b) and (c), plot T versus a and describe the relation between T and a.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 2a(re^{rt}) + (a^2 + 1)(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2ar + (a^2 + 1) = 0$$

$$r = \frac{-2a \pm \sqrt{4a^2 - 4(1)(a^2 + 1)}}{2(1)} = \frac{-2a \pm \sqrt{-4}}{2} = \frac{-2a \pm 2i}{2} = -a \pm i$$

$$r = \{-a - i, -a + i\}$$

Two solutions to the ODE are $y = e^{(-a-i)t}$ and $y = e^{(-a+i)t}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_1 e^{(-a-i)t} + C_2 e^{(-a+i)t} \\
&= C_1 e^{-at-it} + C_2 e^{-at+it} \\
&= C_1 e^{-at} e^{-it} + C_2 e^{-at} e^{it} \\
&= C_1 e^{-at}[\cos(-t) + i\sin(-t)] + C_2 e^{-at}[\cos(t) + i\sin(t)] \\
&= C_1 e^{-at}(\cos t - i\sin t) + C_2 e^{-at}(\cos t + i\sin t) \\
&= C_1 e^{-at}\cos t - iC_1 e^{-at}\sin t + C_2 e^{-at}\cos t + iC_2 e^{-at}\sin t \\
&= (C_1 + C_2)e^{-at}\cos t + (-iC_1 + iC_2)e^{-at}\sin t
\end{aligned}$$

Using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(t) = C_3 e^{-at}\cos t + C_4 e^{-at}\sin t.$$

Take a derivative of it.

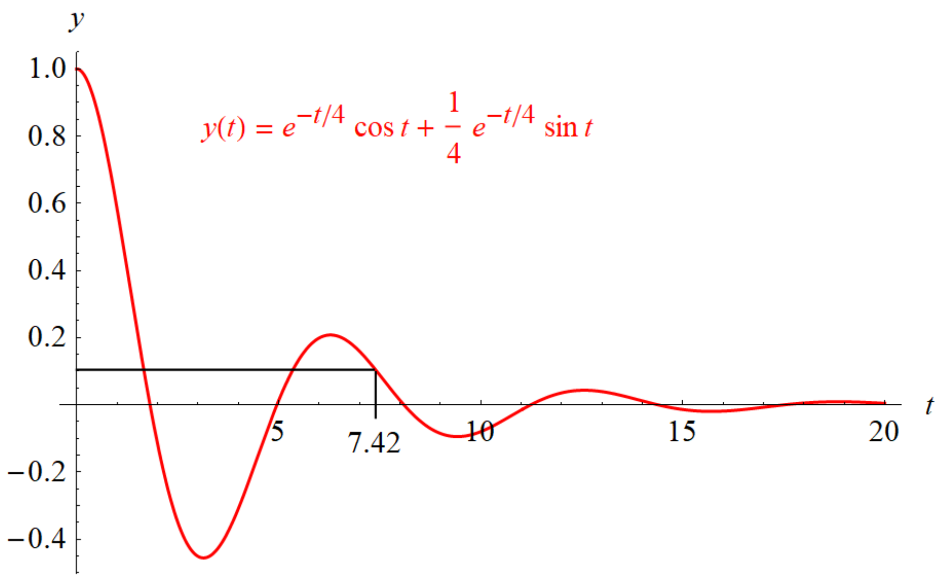
$$y'(t) = -C_3 a e^{-at}\cos t - C_3 e^{-at}\sin t - C_4 a e^{-at}\sin t + C_4 e^{-at}\cos t$$

Apply the initial conditions now to determine C_3 and C_4.

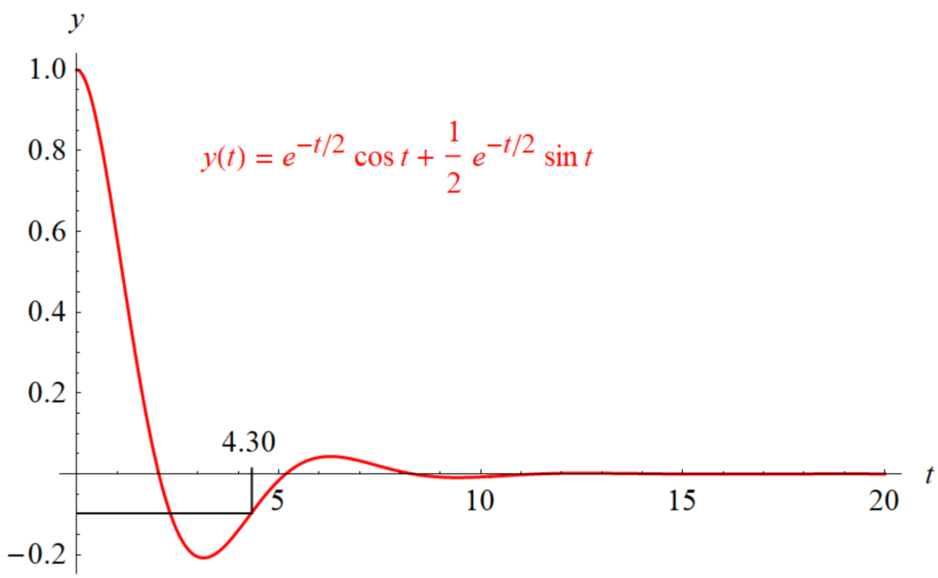
$$y(0) = C_3 = 1$$
$$y'(0) = -C_3 a + C_4 = 0$$

Solving this system of equations yields $C_3 = 1$ and $C_4 = a$. Therefore,

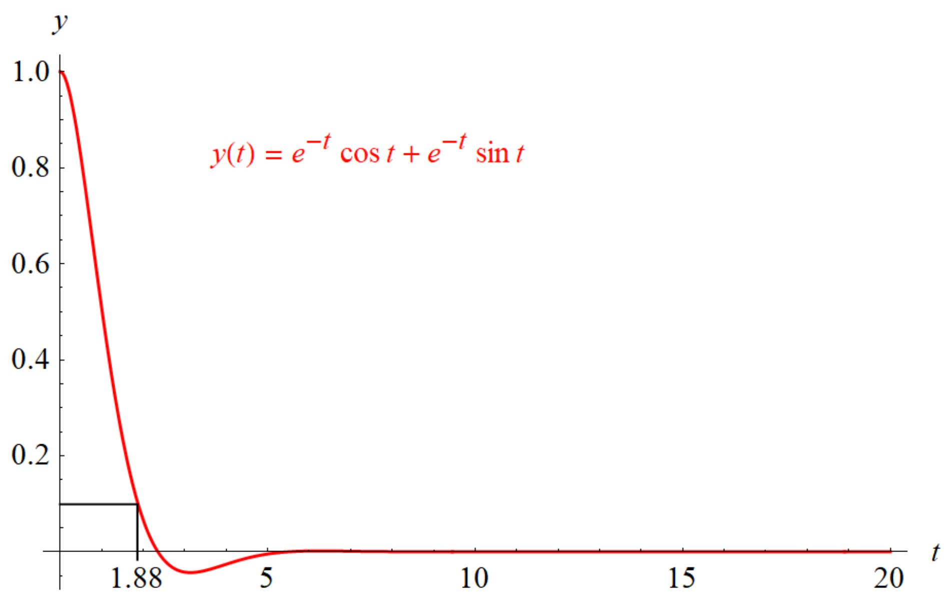
$$y(t) = e^{-at} \cos t + a e^{-at} \sin t.$$



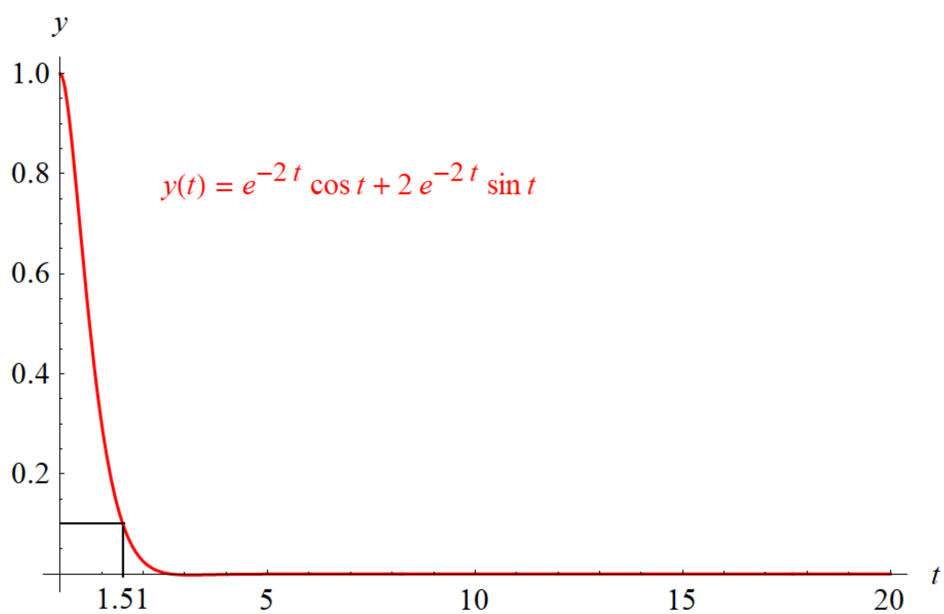
Based on the graph for $a = 1/4$, the smallest T such that $|y(t)| < 0.1$ for $t > T$ is about 7.43.



Based on the graph for $a = 1/2$, the smallest T such that $|y(t)| < 0.1$ for $t > T$ is about 4.30.

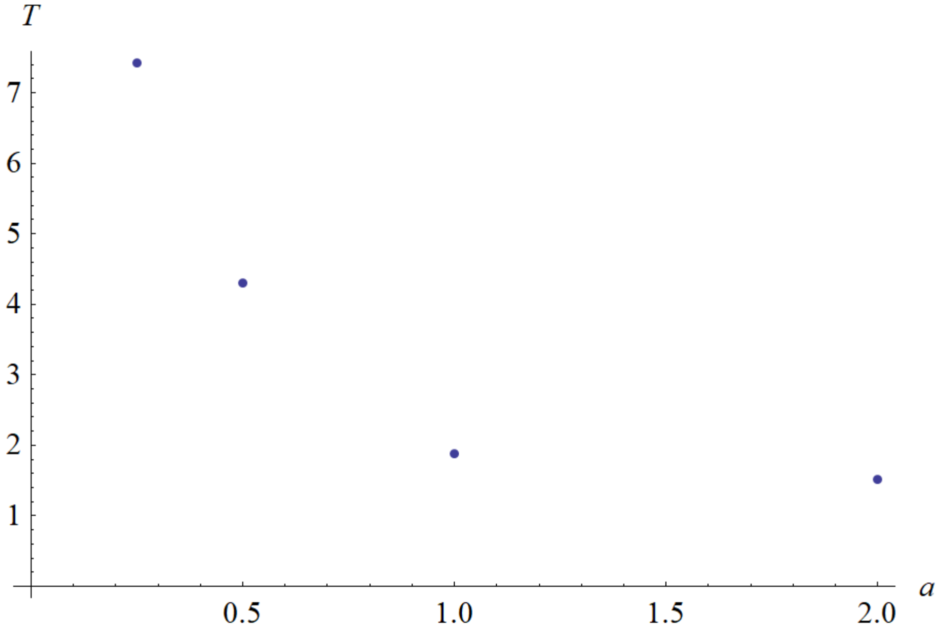


Based on the graph for $a = 1$, the smallest T such that $|y(t)| < 0.1$ for $t > T$ is about 1.88.



Based on the graph for $a = 2$, the smallest T such that $|y(t)| < 0.1$ for $t > T$ is about 1.51.

Plotting the points, $(0.25, 7.43)$, $(0.5, 4.30)$, $(1, 1.88)$, and $(2, 1.51)$, we obtain the following graph.



It seems that T and a are inversely related.