Problem 28

In this problem we outline a different derivation of Euler's formula.

(a) Show that $y_1(t) = \cos t$ and $y_2(t) = \sin t$ are a fundamental set of solutions of $y'' + y = 0$; that is, show that they are solutions and that their Wronskian is not zero.

(b) Show (formally) that $y = e^{it}$ is also a solution of $y'' + y = 0$. Therefore,

$$e^{it} = c_1 \cos t + c_2 \sin t \tag{i}$$

for some constants c_1 and c_2. Why is this so?

(c) Set $t = 0$ in Eq. (i) to show that $c_1 = 1$.

(d) Assuming that Eq. (14) is true, differentiate Eq. (i) and then set $t = 0$ to conclude that $c_2 = i$. Use the values of c_1 and c_2 in Eq. (i) to arrive at Euler's formula.

Solution

Part (a)

Check that $y_1(t) = \cos t$ is a solution of the ODE.

$$y_1'' + y_1 \stackrel{?}{=} 0$$

$$\frac{d^2}{dt^2}(\cos t) + (\cos t) \stackrel{?}{=} 0$$

$$-\cos t + \cos t \stackrel{?}{=} 0$$

$$0 = 0$$

Now check that $y_2(t) = \sin t$ is a solution of the ODE.

$$y_2'' + y_2 \stackrel{?}{=} 0$$

$$\frac{d^2}{dt^2}(\sin t) + (\sin t) \stackrel{?}{=} 0$$

$$-\sin t + \sin t \stackrel{?}{=} 0$$

$$0 = 0$$

Now calculate $W(y_1, y_2)$, the Wronskian of y_1 and y_2.

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} \cos t & \sin t \\ \frac{d}{dt}(\cos t) & \frac{d}{dt}(\sin t) \end{vmatrix} \\ &= \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} \\ &= \cos t(\cos t) - \sin t(-\sin t) \\ &= \cos^2 t + \sin^2 t \\ &= 1 \end{aligned}$$

Because the Wronskian of y_1 and y_2 is nonzero, y_1 and y_2 form a fundamental set of solutions.

Part (b)

$$y'' + y = 0$$

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 1 = 0$$

$$r = \{-i, i\}$$

Therefore, two solutions to the ODE are $y = e^{-it}$ and $y = e^{it}$. Since $\cos t$ and $\sin t$ form a fundamental set of solutions, e^{it} can be written as a linear combination of the two for certain constants, c_1 and c_2.

$$e^{it} = c_1 \cos t + c_2 \sin t \tag{i}$$

Part (c)

These constants must hold for all values of t, so one of them can be determined by setting $t = 0$.

$$e^0 = c_1 \cos 0 + c_2 \sin 0 \quad \rightarrow \quad c_1 = 1$$

Part (d)

To determine c_2, differentiate both sides of equation (i) with respect to t.

$$ie^{it} = -c_1 \sin t + c_2 \cos t$$

Set $t = 0$ again to determine the second constant.

$$ie^0 = -c_1 \sin 0 + c_2 \cos 0 \quad \rightarrow \quad c_2 = i$$

Therefore, we obtain Euler's formula.

$$e^{it} = \cos t + i \sin t$$