Problem 32

Consider the differential equation

$$ay'' + by' + cy = 0,$$

where $b^2 - 4ac < 0$ and the characteristic equation has complex roots $\lambda \pm i\mu$. Substitute the functions

$$u(t) = e^{\lambda t}\cos\mu t \quad \text{and} \quad v(t) = e^{\lambda t}\sin\mu t$$

for y in the differential equation and thereby confirm that they are solutions.

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$a(r^2 e^{rt}) + b(re^{rt}) + c(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$ar^2 + br + c = 0$$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} = -\frac{b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}$$

We see that

$$\lambda = -\frac{b}{2a}$$

$$\mu = \frac{\sqrt{4ac - b^2}}{2a}.$$

Now check that $u(t) = e^{\lambda t}\cos\mu t$ is a solution to the ODE.

$$au'' + bu' + cu \stackrel{?}{=} 0$$

$$a\frac{d^2}{dt^2}(e^{\lambda t}\cos\mu t) + b\frac{d}{dt}(e^{\lambda t}\cos\mu t) + c(e^{\lambda t}\cos\mu t) \stackrel{?}{=} 0$$

$$a\frac{d}{dt}(\lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t) + b(\lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t) + ce^{\lambda t}\cos\mu t \stackrel{?}{=} 0$$

$$a(\lambda^2 e^{\lambda t}\cos\mu t - \mu\lambda e^{\lambda t}\sin\mu t - \mu\lambda e^{\lambda t}\sin\mu t - \mu^2 e^{\lambda t}\cos\mu t) + b(\lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t) + ce^{\lambda t}\cos\mu t \stackrel{?}{=} 0$$

$$a\lambda^2 e^{\lambda t}\cos\mu t - a\mu\lambda e^{\lambda t}\sin\mu t - a\mu\lambda e^{\lambda t}\sin\mu t - a\mu^2 e^{\lambda t}\cos\mu t + b\lambda e^{\lambda t}\cos\mu t - b\mu e^{\lambda t}\sin\mu t + ce^{\lambda t}\cos\mu t \stackrel{?}{=} 0$$

$$e^{\lambda t}(a\lambda^2 - a\mu^2 + b\lambda + c)\cos\mu t - \mu e^{\lambda t}(2a\lambda + b)\sin\mu t \stackrel{?}{=} 0$$

$$e^{\lambda t}(0)\cos\mu t - \mu e^{\lambda t}(0)\sin\mu t \stackrel{?}{=} 0$$

$$0 = 0$$

Now check that $v(t) = e^{\lambda t}\sin\mu t$ is a solution to the ODE.

$$av'' + bv' + cv \stackrel{?}{=} 0$$

$$a\frac{d^2}{dt^2}(e^{\lambda t}\sin\mu t) + b\frac{d}{dt}(e^{\lambda t}\sin\mu t) + c(e^{\lambda t}\sin\mu t) \stackrel{?}{=} 0$$

$$a\frac{d}{dt}(\lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t) + b(\lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t) + ce^{\lambda t}\sin\mu t \stackrel{?}{=} 0$$

$$a(\lambda^2 e^{\lambda t}\sin\mu t + \mu\lambda e^{\lambda t}\cos\mu t + \mu\lambda e^{\lambda t}\cos\mu t - \mu^2 e^{\lambda t}\sin\mu t) + b(\lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t) + ce^{\lambda t}\sin\mu t \stackrel{?}{=} 0$$

$$a\lambda^2 e^{\lambda t}\sin\mu t + a\mu\lambda e^{\lambda t}\cos\mu t + a\mu\lambda e^{\lambda t}\cos\mu t - a\mu^2 e^{\lambda t}\sin\mu t + b\lambda e^{\lambda t}\sin\mu t + b\mu e^{\lambda t}\cos\mu t + ce^{\lambda t}\sin\mu t \stackrel{?}{=} 0$$

$$\mu e^{\lambda t}(2a\lambda + b)\cos\mu t + e^{\lambda t}(a\lambda^2 - a\mu^2 + b\lambda + c)\sin\mu t \stackrel{?}{=} 0$$

$$\mu e^{\lambda t}(0)\cos\mu t + e^{\lambda t}(0)\sin\mu t \stackrel{?}{=} 0$$

$$0 = 0$$