Problem 34

Euler Equations. An equation of the form

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0, \qquad t > 0, \tag{ii}$$

where α and β are real constants, is called an Euler equation.

(a) Let $x = \ln t$ and calculate dy/dt and d^2y/dt^2 in terms of dy/dx and d^2y/dx^2.

(b) Use the results of part (a) to transform Eq. (ii) into

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0. \tag{iii}$$

Observe that Eq. (iii) has constant coefficients. If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of Eq. (iii), then $y_1(\ln t)$ and $y_2(\ln t)$ form a fundamental set of solutions of Eq. (ii).

Solution

Make the substitution $x = \ln t$ in the ODE. Then

$$e^x = t \quad \to \quad e^{2x} = t^2,$$

and the ODE becomes

$$e^{2x}\frac{d^2y}{dt^2} + \alpha e^x\frac{dy}{dt} + \beta y = 0.$$

The aim now is to find what the derivatives are in terms of this new variable by using the chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\left(\frac{1}{t}\right) = \frac{dy}{dx}\left(\frac{1}{e^x}\right) = e^{-x}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right) = \frac{1}{t}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) = \frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right)$$

Substitute these expressions into the ODE.

$$e^{2x}\frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + \alpha e^x\left(e^{-x}\frac{dy}{dx}\right) + \beta y = 0$$

$$e^x\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + \alpha\frac{dy}{dx} + \beta y = 0$$

$$-\frac{dy}{dx} + \frac{d^2y}{dx^2} + \alpha\frac{dy}{dx} + \beta y = 0$$

Therefore,

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0.$$