Problem 36

In each of Problems 35 through 42, use the method of Problem 34 to solve the given equation for $t > 0$.

$$t^2y'' + 4ty' + 2y = 0$$

Solution

The Hard Way

Make the substitution $x = \ln t$ in the ODE. Then

$$e^x = t \quad \rightarrow \quad e^{2x} = t^2,$$

and the ODE becomes

$$e^{2x}\frac{d^2y}{dt^2} + 4e^x\frac{dy}{dt} + 2y = 0.$$

The aim now is to find what the derivatives are in terms of this new variable by using the chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{dy}{dx} \left( \frac{1}{t} \right) = \frac{dy}{dx} \left( \frac{1}{e^x} \right) = e^{-x}\frac{dy}{dx},$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = \frac{dx}{dt} \frac{d}{dx} \left( e^{-x}\frac{dy}{dx} \right) = \frac{1}{t} \left( -e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2} \right) = \frac{1}{e^x} \left( -e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2} \right).$$

Substitute these expressions into the ODE.

$$e^{2x} \frac{1}{e^x} \left( -e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2} \right) + 4e^x \left( e^{-x}\frac{dy}{dx} \right) + 2y = 0

\quad \Rightarrow \quad e^x \left( -e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2} \right) + 4\frac{dy}{dx} + 2y = 0

\quad \Rightarrow \quad \frac{dy}{dx} + 4\frac{dy}{dx} + 2y = 0

\quad \Rightarrow \quad \frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0 \quad (1)$$

The transformed ODE is one with constant coefficients, so its solution is of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Substitute these expressions into equation (1).

$$r^2e^{rx} + 3(re^{rx}) + 2(e^{rx}) = 0$$

Divide both sides by $e^{rx}$.

$$r^2 + 3r + 2 = 0$$

$$(r + 2)(r + 1) = 0$$

$$r = \{-2, -1\}$$
Two solutions to the ODE are \( y = e^{-2x} \) and \( y = e^{-x} \), so the general solution is a linear combination of the two.

\[
y(x) = C_1 e^{-2x} + C_2 e^{-x}
\]

Change back to the original variable now.

\[
y(t) = C_1 e^{-2 \ln t} + C_2 e^{-\ln t} = C_1 t^{-2} + C_2 t^{-1}.
\]

Therefore,

\[
y(t) = C_1 t^{-2} + C_2 t^{-1}.
\]

**The Easy Way**

\[
t^2 y'' + 4ty' + 2y = 0
\]

Since this is an Euler (or equidimensional) equation, the solution is of the form \( y = t^r \).

\[
y = t^r \quad \rightarrow \quad y' = rt^{r-1} \quad \rightarrow \quad y'' = r(r-1)t^{r-2}
\]

Substitute these expressions into the ODE.

\[
t^2[r(r-1)t^{r-2}] + 4t(rt^{r-1}) + 2t^r = 0
\]

\[
r(r-1)t^r + 4rt^r + 2t^r = 0
\]

Divide both sides by \( t^r \).

\[
r(r-1) + 4r + 2 = 0
\]

\[
r^2 + 3r + 2 = 0
\]

\[
(r + 2)(r + 1) = 0
\]

\[
r = \{-2, -1\}
\]

Two solutions to the ODE are \( y = t^{-2} \) and \( y = t^{-1} \), so the general solution is

\[
y(t) = C_3 t^{-2} + C_4 t^{-1},
\]

a linear combination of the two.

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