Problem 36

In each of Problems 35 through 42, use the method of Problem 34 to solve the given equation for $t > 0$.

$$t^2y'' + 4ty' + 2y = 0$$

Solution

The Hard Way

Make the substitution $x = \ln t$ in the ODE. Then

$$e^x = t \quad \to \quad e^{2x} = t^2,$$

and the ODE becomes

$$e^{2x}\frac{d^2y}{dt^2} + 4e^x\frac{dy}{dt} + 2y = 0.$$

The aim now is to find what the derivatives are in terms of this new variable by using the chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\left(\frac{1}{t}\right) = \frac{dy}{dx}\left(\frac{1}{e^x}\right) = e^{-x}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right) = \frac{1}{t}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) = \frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right)$$

Substitute these expressions into the ODE.

$$e^{2x}\frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + 4e^x\left(e^{-x}\frac{dy}{dx}\right) + 2y = 0$$

$$e^x\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) + 4\frac{dy}{dx} + 2y = 0$$

$$-\frac{dy}{dx} + \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 2y = 0$$

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0 \tag{1}$$

The transformed ODE is one with constant coefficients, so its solution is of the form $y = e^{rx}$.

$$y = e^{rx} \quad \to \quad \frac{dy}{dx} = re^{rx} \quad \to \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Substitute these expressions into equation (1).

$$r^2e^{rx} + 3(re^{rx}) + 2(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 + 3r + 2 = 0$$

$$(r+2)(r+1) = 0$$

$$r = \{-2, -1\}$$

Two solutions to the ODE are $y = e^{-2x}$ and $y = e^{-x}$, so the general solution is a linear combination of the two.

$$y(x) = C_1 e^{-2x} + C_2 e^{-x}$$

Change back to the original variable now.

$$\begin{aligned} y(t) &= C_1 e^{-2\ln t} + C_2 e^{-\ln t} \\ &= C_1 e^{\ln t^{-2}} + C_2 e^{\ln t^{-1}} \end{aligned}$$

Therefore,

$$y(t) = C_1 t^{-2} + C_2 t^{-1}.$$

The Easy Way

$$t^2 y'' + 4ty' + 2y = 0$$

Since this is an Euler (or equidimensional) equation, the solution is of the form $y = t^r$.

$$y = t^r \quad \to \quad y' = rt^{r-1} \quad \to \quad y'' = r(r-1)t^{r-2}$$

Substitute these expressions into the ODE.

$$t^2[r(r-1)t^{r-2}] + 4t(rt^{r-1}) + 2t^r = 0$$

$$r(r-1)t^r + 4rt^r + 2t^r = 0$$

Divide both sides by t^r.

$$r(r-1) + 4r + 2 = 0$$

$$r^2 + 3r + 2 = 0$$

$$(r+2)(r+1) = 0$$

$$r = \{-2, -1\}$$

Two solutions to the ODE are $y = t^{-2}$ and $y = t^{-1}$, so the general solution is

$$y(t) = C_3 t^{-2} + C_4 t^{-1},$$

a linear combination of the two.