Problem 40

In each of Problems 35 through 42, use the method of Problem 34 to solve the given equation for $t > 0$.

$$t^2y'' - ty' + 5y = 0$$

Solution

The Hard Way

Make the substitution $x = \ln t$ in the ODE. Then

$$e^x = t \quad \rightarrow \quad e^{2x} = t^2,$$

and the ODE becomes

$$e^{2x}\frac{d^2y}{dt^2} - e^x\frac{dy}{dt} + 5y = 0.$$

The aim now is to find what the derivatives are in terms of this new variable by using the chain rule.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\left(\frac{1}{t}\right) = \frac{dy}{dx}\left(\frac{1}{e^x}\right) = e^{-x}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(e^{-x}\frac{dy}{dx}\right) = \frac{1}{t}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) = \frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right)$$

Substitute these expressions into the ODE.

$$e^{2x}\frac{1}{e^x}\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) - e^x\left(e^{-x}\frac{dy}{dx}\right) + 5y = 0$$

$$e^x\left(-e^{-x}\frac{dy}{dx} + e^{-x}\frac{d^2y}{dx^2}\right) - \frac{dy}{dx} + 5y = 0$$

$$-\frac{dy}{dx} + \frac{d^2y}{dx^2} - \frac{dy}{dx} + 5y = 0$$

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0 \tag{1}$$

The transformed ODE is one with constant coefficients, so its solution is of the form $y = e^{rx}$.

$$y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2e^{rx}$$

Substitute these expressions into equation (1).

$$r^2e^{rx} - 2(re^{rx}) + 5(e^{rx}) = 0$$

Divide both sides by e^{rx}.

$$r^2 - 2r + 5 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$$r = \{1 - 2i, 1 + 2i\}$$

Two solutions to the ODE are $y = e^{(1-2i)x}$ and $y = e^{(1+2i)x}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(x) &= C_1 e^{(1-2i)x} + C_2 e^{(1+2i)x} \\
&= C_1 e^{x-2ix} + C_2 e^{x+2ix} \\
&= C_1 e^x e^{-2ix} + C_2 e^x e^{2ix} \\
&= C_1 e^x[\cos(-2x) + i\sin(-2x)] + C_2 e^x[\cos(2x) + i\sin(2x)] \\
&= C_1 e^x[\cos(2x) - i\sin(2x)] + C_2 e^x[\cos(2x) + i\sin(2x)] \\
&= C_1 e^x \cos 2x - iC_1 e^x \sin 2x + C_2 e^x \cos 2x + iC_2 e^x \sin 2x \\
&= (C_1 + C_2)e^x \cos 2x + (-iC_1 + iC_2)e^x \sin 2x
\end{aligned}$$

Using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$, the real general solution is

$$y(x) = C_3 e^x \cos 2x + C_4 e^x \sin 2x.$$

Change back to the original variable now.

$$y(t) = C_3 e^{\ln t} \cos(2\ln t) + C_4 e^{\ln t} \sin(2\ln t)$$

Therefore,

$$y(t) = C_3 t\cos(2\ln t) + C_4 t\sin(2\ln t).$$

The Easy Way

$$t^2 y'' - ty' + 5y = 0$$

Since this is an Euler (or equidimensional) equation, the solution is of the form $y = t^r$.

$$y = t^r \quad \rightarrow \quad y' = rt^{r-1} \quad \rightarrow \quad y'' = r(r-1)t^{r-2}$$

Substitute these expressions into the ODE.

$$t^2[r(r-1)t^{r-2}] - t(rt^{r-1}) + 5t^r = 0$$

$$r(r-1)t^r - rt^r + 5t^r = 0$$

Divide both sides by t^r.

$$r(r-1) - r + 5 = 0$$

$$r^2 - 2r + 5 = 0$$

$$r = \frac{2 \pm \sqrt{4 - 4(1)(5)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i$$

$$r = \{1 - 2i, 1 + 2i\}$$

Two solutions to the ODE are $y = t^{1-2i}$ and $y = t^{1+2i}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_5 t^{1-2i} + C_6 t^{1+2i} \\
&= C_5 t^1 t^{-2i} + C_6 t^1 t^{2i} \\
&= C_5 t e^{\ln t^{-2i}} + C_6 t e^{\ln t^{2i}} \\
&= C_5 t e^{(-2i)\ln t} + C_6 t e^{(2i)\ln t} \\
&= C_5 t[\cos(-2\ln t) + i\sin(-2\ln t)] + C_6 t[\cos(2\ln t) + i\sin(2\ln t)] \\
&= C_5 t[\cos(2\ln t) - i\sin(2\ln t)] + C_6 t[\cos(2\ln t) + i\sin(2\ln t)] \\
&= C_5 t\cos(2\ln t) - iC_5 t\sin(2\ln t) + C_6 t\cos(2\ln t) + iC_6 t\sin(2\ln t) \\
&= (C_5 + C_6)t\cos(2\ln t) + (-iC_5 + iC_6)t\sin(2\ln t)
\end{aligned}$$

Using C_7 for $C_5 + C_6$ and C_8 for $-iC_5 + iC_6$, the real general solution is

$$y(t) = C_7 t\cos(2\ln t) + C_8 t\sin(2\ln t).$$