Problem 41

In each of Problems 35 through 42, use the method of Problem 34 to solve the given equation for \( t > 0 \).

\[
t^2y'' + 3ty' - 3y = 0
\]

Solution

The Hard Way

Make the substitution \( x = \ln t \) in the ODE. Then

\[
e^x = t \quad \rightarrow \quad e^{2x} = t^2,
\]

and the ODE becomes

\[
e^{2x} \frac{d^2y}{dt^2} + 3e^x \frac{dy}{dt} - 3y = 0.
\]

The aim now is to find what the derivatives are in terms of this new variable by using the chain rule.

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} = \frac{dy}{dx} \frac{1}{t} = \frac{1}{e^x} \frac{dy}{dx},
\]

\[
\frac{d^2y}{dt^2} = \frac{d}{dt} \left( \frac{dy}{dt} \right) = \frac{dx}{dt} \frac{d}{dx} \left( e^{-x} \frac{dy}{dx} \right) = \frac{1}{t} \left( -e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2y}{dx^2} \right) = \frac{1}{e^x} \left( -e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2y}{dx^2} \right)
\]

Substitute these expressions into the ODE.

\[
e^{2x} \frac{1}{e^x} \left( -e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2y}{dx^2} \right) + 3e^x \left( e^{-x} \frac{dy}{dx} \right) - 3y = 0
\]

\[
e^x \left( -e^{-x} \frac{dy}{dx} + e^{-x} \frac{d^2y}{dx^2} \right) + 3 \frac{dy}{dx} - 3y = 0
\]

\[
- \frac{dy}{dx} + \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} - 3y = 0
\]

\[
\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} - 3y = 0 \quad (1)
\]

The transformed ODE is one with constant coefficients, so its solution is of the form \( y = e^{rx} \).

\[
y = e^{rx} \quad \rightarrow \quad \frac{dy}{dx} = re^{rx} \quad \rightarrow \quad \frac{d^2y}{dx^2} = r^2 e^{rx}
\]

Substitute these expressions into equation (1).

\[
r^2 e^{rx} + 2(re^{rx}) - 3(e^{rx}) = 0
\]

Divide both sides by \( e^{rx} \).

\[
r^2 + 2r - 3 = 0
\]

\[
(r + 3)(r - 1) = 0
\]

\[
r = \{-3, 1\}
\]

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Two solutions to the ODE are $y = e^{-3x}$ and $y = e^x$, so the general solution is a linear combination of the two.

$$y(x) = C_1e^{-3x} + C_2e^x$$

Change back to the original variable now.

$$y(t) = C_1e^{-3\ln t} + C_2e^{\ln t} = C_1t^{-3} + C_2t$$

Therefore,

$$y(t) = C_1t^{-3} + C_2t.$$  

**The Easy Way**

$$t^2y'' + 3ty' - 3y = 0$$

Since this is an Euler (or equidimensional) equation, the solution is of the form $y = t^r$.

$$y = t^r \to y' = rt^{r-1} \to y'' = r(r-1)t^{r-2}$$

Substitute these expressions into the ODE.

$$t^2[r(r-1)t^{r-2}] + 3t(rt^{r-1}) - 3t^r = 0$$

$$r(r-1)t^r + 3rt^r - 3t^r = 0$$

Divide both sides by $t^r$.

$$r(r-1) + 3r - 3 = 0$$

$$r^2 + 2r - 3 = 0$$

$$(r + 3)(r - 1) = 0$$

$$r = \{-3, 1\}$$

Two solutions to the ODE are $y = t^{-3}$ and $y = t^1$, so the general solution is a linear combination of the two.

$$y(t) = C_3t^{-3} + C_4t$$