Problem 43

In this problem we determine conditions on p and q that enable Eq. (i) to be transformed into an equation with constant coefficients by a change of the independent variable. Let $x = u(t)$ be the new independent variable, with the relation between x and t to be specified later.

(a) Show that

$$\frac{dy}{dt} = \frac{dx}{dt}\frac{dy}{dx}, \qquad \frac{d^2y}{dt^2} = \left(\frac{dx}{dt}\right)^2 \frac{d^2y}{dx^2} + \frac{d^2x}{dt^2}\frac{dy}{dx}.$$

(b) Show that the differential equation (i) becomes

$$\left(\frac{dx}{dt}\right)^2 \frac{d^2y}{dx^2} + \left(\frac{d^2x}{dt^2} + p(t)\frac{dx}{dt}\right)\frac{dy}{dx} + q(t)y = 0. \tag{iv}$$

(c) In order for Eq. (iv) to have constant coefficients, the coefficients of d^2y/dx^2 and of y must be proportional. If $q(t) > 0$, then we can choose the constant of proportionality to be 1; hence

$$x = u(t) = \int [q(t)]^{1/2}\, dt. \tag{v}$$

(d) With x chosen as in part (c), show that the coefficient of dy/dx in Eq. (iv) is also a constant, provided that the expression

$$\frac{q'(t) + 2p(t)q(t)}{2[q(t)]^{3/2}} \tag{vi}$$

is a constant. Thus Eq. (i) can be transformed into an equation with constant coefficients by a change of the independent variable, provided that the function $(q' + 2pq)/q^{3/2}$ is a constant. How must this result be modified if $q(t) < 0$?

Solution

Eq. (i) is

$$y'' + p(t)y' + q(t)y = 0. \tag{i}$$

The aim is to use a change of variables, $x = u(t)$, to turn this into a constant-coefficient ODE. By the chain rule,

$$\begin{aligned}
\frac{dy}{dt} &= \frac{dy}{dx}\frac{dx}{dt} \\
\frac{d^2y}{dt^2} &= \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\frac{dx}{dt}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dx}{dt} + \frac{dy}{dx}\frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{dx}{dt}\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt} + \frac{dy}{dx}\frac{d^2x}{dt^2} \\
&= \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2}.
\end{aligned}$$

Substitute these expressions for the derivatives into Eq. (i).

$$\left[\frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2}\right] + p(t)\left(\frac{dy}{dx}\frac{dx}{dt}\right) + q(t)y = 0$$

$$\left(\frac{dx}{dt}\right)^2\frac{d^2y}{dx^2}+\frac{d^2x}{dt^2}\frac{dy}{dx}+p(t)\frac{dx}{dt}\frac{dy}{dx}+q(t)y=0$$

Therefore,

$$\left(\frac{dx}{dt}\right)^2\frac{d^2y}{dx^2}+\left(\frac{d^2x}{dt^2}+p(t)\frac{dx}{dt}\right)\frac{dy}{dx}+q(t)y=0. \tag{iv}$$

In order for Eq. (iv) to have constant coefficients, the coefficients of d^2y/dx^2 and y must be proportional.

$$\left(\frac{dx}{dt}\right)^2\propto q(t)$$

Introduce a proportionality constant k to change this to an equation we can use.

$$\left(\frac{dx}{dt}\right)^2=kq(t)$$

Take the square root of both sides.

$$\frac{dx}{dt}=\pm[kq(t)]^{1/2} \tag{1}$$

Differentiate both sides with respect to t.

$$\begin{aligned}\frac{d^2x}{dt^2}&=\pm\frac{1}{2}[kq(t)]^{-1/2}kq'(t)\\&=\pm\frac{kq'(t)}{2[kq(t)]^{1/2}}\end{aligned}$$

Now substitute the previous three equations into Eq. (iv).

$$kq(t)\frac{d^2y}{dx^2}+\left(\pm\frac{kq'(t)}{2[kq(t)]^{1/2}}\pm p(t)[kq(t)]^{1/2}\right)\frac{dy}{dx}+q(t)y=0$$

$$kq(t)\frac{d^2y}{dx^2}\pm\left(\frac{kq'(t)+2p(t)[kq(t)]}{2[kq(t)]^{1/2}}\right)\frac{dy}{dx}+q(t)y=0$$

Divide both sides by $kq(t)$.

$$\frac{d^2y}{dx^2}\pm\left(\frac{kq'(t)+2p(t)[kq(t)]}{2[kq(t)]^{3/2}}\right)\frac{dy}{dx}+\frac{1}{k}y=0$$

Provided that the coefficient of dy/dx is constant,

$$\pm\left(\frac{kq'(t)+2p(t)[kq(t)]}{2[kq(t)]^{3/2}}\right)=A,\quad\text{or}\quad\frac{q'(t)+2p(t)q(t)}{[kq(t)]^{3/2}}=\pm\frac{2A}{k}=B$$

if we multiply both sides by $\pm 2/k$, Eq. (i) can be transformed into a constant-coefficient ODE by $x=u(t)$. Integrating both sides of equation (1) with respect to t, we find what x is exactly.

$$x(t)=\pm\int^t[kq(s)]^{1/2}\,ds$$

Note that for $[kq(s)]^{1/2}$ and $[kq(t)]^{3/2}$ to be real numbers, k should be chosen to have the same sign as $q(t)$. For example, if $q(t)$ is positive, k can be chosen to be 1 for convenience. Or if $q(t)$ is negative, k can be chosen to be -1.