Problem 1

In each of Problems 1 through 10, find the general solution of the given differential equation.

$$y'' - 2y' + y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} - 2(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = \{1\}$$

One solution to the ODE is $y = e^t$. Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, $y = ce^t$. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t.

$$y(t) = c(t)e^t$$

Substitute this expression for y into the original ODE to determine $c(t)$.

$$y'' - 2y' + y = 0 \quad \rightarrow \quad [c(t)e^t]'' - 2[c(t)e^t]' + [c(t)e^t] = 0$$

Evaluate the derivatives using the product rule.

$$[c'(t)e^t + c(t)e^t]' - 2[c'(t)e^t + c(t)e^t] + c(t)e^t = 0$$

$$c''(t)e^t + \cancel{c'(t)e^t} + \cancel{c'(t)e^t} + \cancel{c(t)e^t} - \cancel{2c'(t)e^t} - \cancel{2c(t)e^t} + \cancel{c(t)e^t} = 0$$

$$c''(t)e^t = 0$$

Divide both sides by e^t.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1t + C_2$$

Therefore, the general solution is

$$y(t) = C_1te^t + C_2e^t.$$