Problem 4

In each of Problems 1 through 10, find the general solution of the given differential equation.

$$4y'' + 12y' + 9y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$4(r^2e^{rt}) + 12(re^{rt}) + 9(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$4r^2 + 12r + 9 = 0$$

$$(2r+3)^2 = 0$$

$$r = \left\{-\frac{3}{2}\right\}$$

One solution to the ODE is $y = e^{-3t/2}$. Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, $y = ce^{-3t/2}$. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t.

$$y(t) = c(t)e^{-3t/2}$$

Substitute this expression for y into the original ODE to determine $c(t)$.

$$4y'' + 12y' + 9y = 0 \quad \rightarrow \quad 4[c(t)e^{-3t/2}]'' + 12[c(t)e^{-3t/2}]' + 9[c(t)e^{-3t/2}] = 0$$

Evaluate the derivatives using the product rule.

$$4\left[c'(t)e^{-3t/2} - \frac{3c(t)}{2}e^{-3t/2}\right]' + 12\left[c'(t)e^{-3t/2} - \frac{3c(t)}{2}e^{-3t/2}\right] + 9c(t)e^{-3t/2} = 0$$

$$4\left[c''(t)e^{-3t/2} - \frac{3c'(t)}{2}e^{-3t/2} - \frac{3c'(t)}{2}e^{-3t/2} + \frac{9c(t)}{4}e^{-3t/2}\right] + 12\left[c'(t)e^{-3t/2} - \frac{3c(t)}{2}e^{-3t/2}\right] + 9c(t)e^{-3t/2} = 0$$

$$4c''(t)e^{-3t/2} - \cancel{6c'(t)e^{-3t/2}} - \cancel{6c'(t)e^{-3t/2}} + \cancel{9c(t)e^{-3t/2}} + \cancel{12c'(t)e^{-3t/2}} - \cancel{18c(t)e^{-3t/2}} + \cancel{9c(t)e^{-3t/2}} = 0$$

$$4c''(t)e^{-3t/2} = 0$$

Divide both sides by $4e^{-3t/2}$.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1t + C_2$$

Therefore, the general solution is

$$y(t) = C_1te^{-3t/2} + C_2e^{-3t/2}.$$