Problem 4

In each of Problems 1 through 10, find the general solution of the given differential equation.

\[ 4y'' + 12y' + 9y = 0 \]

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form \( y = e^{rt} \).

\[ y = e^{rt} \rightarrow y' = re^{rt} \rightarrow y'' = r^2e^{rt} \]

Substitute these expressions into the ODE.

\[ 4(r^2e^{rt}) + 12(re^{rt}) + 9(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ 4r^2 + 12r + 9 = 0 \]

\[ (2r + 3)^2 = 0 \]

\[ r = \left\{ \frac{-3}{2} \right\} \]

One solution to the ODE is \( y = e^{-3t/2} \). Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, \( y = ce^{-3t/2} \). According to the method of reduction of order, the general solution is found by allowing \( c \) to vary as a function of \( t \).

\[ y(t) = c(t)e^{-3t/2} \]

Substitute this expression for \( y \) into the original ODE to determine \( c(t) \).

\[ 4y'' + 12y' + 9y = 0 \quad \rightarrow \quad 4[c(t)e^{-3t/2}]'' + 12[c(t)e^{-3t/2}]' + 9[c(t)e^{-3t/2}] = 0 \]

Evaluate the derivatives using the product rule.

\[ 4 \left[ c'(t)e^{-3t/2} - \frac{3c(t)}{2}e^{-3t/2} \right]' + 12 \left[ c'(t)e^{-3t/2} - \frac{3c(t)}{2}e^{-3t/2} \right] + 9c(t)e^{-3t/2} = 0 \]

\[ 4 \left[ c''(t)e^{-3t/2} - \frac{3c'(t)}{2}e^{-3t/2} - \frac{3c(t)}{4}e^{-3t/2} + \frac{9c(t)}{4}e^{-3t/2} \right] + 12 \left[ c'(t)e^{-3t/2} - \frac{3c(t)}{2}e^{-3t/2} \right] + 9c(t)e^{-3t/2} = 0 \]

\[ 4c''(t)e^{-3t/2} - 6c'(t)e^{-3t/2} - 6c'(t)e^{-3t/2} + 9c(t)e^{-3t/2} + 12c'(t)e^{-3t/2} - 18c(t)e^{-3t/2} + 9c(t)e^{-3t/2} = 0 \]

\[ 4c''(t)e^{-3t/2} = 0 \]

Divide both sides by \( 4e^{-3t/2} \).

\[ c''(t) = 0 \]

Integrate both sides with respect to \( t \).

\[ c'(t) = C_1 \]

Integrate both sides with respect to \( t \) once more.

\[ c(t) = C_1t + C_2 \]

Therefore, the general solution is

\[ y(t) = C_1te^{-3t/2} + C_2e^{-3t/2} \]

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