Problem 8

In each of Problems 1 through 10, find the general solution of the given differential equation.

$$16y'' + 24y' + 9y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$16(r^2 e^{rt}) + 24(re^{rt}) + 9(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$16r^2 + 24r + 9 = 0$$

$$(4r+3)^2 = 0$$

$$r = \left\{-\frac{3}{4}\right\}$$

One solution to the ODE is $y = e^{-3t/4}$. Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, $y = ce^{-3t/4}$. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t.

$$y(t) = c(t)e^{-3t/4}$$

Substitute this expression for y into the original ODE to determine $c(t)$.

$$16y'' + 24y' + 9y = 0 \quad \rightarrow \quad 16[c(t)e^{-3t/4}]'' + 24[c(t)e^{-3t/4}]' + 9[c(t)e^{-3t/4}] = 0$$

Evaluate the derivatives using the product rule.

$$16\left[c'(t)e^{-3t/4} - \frac{3}{4}c(t)e^{-3t/4}\right]' + 24\left[c'(t)e^{-3t/4} - \frac{3}{4}c(t)e^{-3t/4}\right] + 9[c(t)e^{-3t/4}] = 0$$

$$16\left[c''(t)e^{-3t/4} - \frac{3}{4}c'(t)e^{-3t/4} - \frac{3}{4}c'(t)e^{-3t/4} + \frac{9}{16}c(t)e^{-3t/4}\right] + 24\left[c'(t)e^{-3t/4} - \frac{3}{4}c(t)e^{-3t/4}\right] + 9[c(t)e^{-3t/4}] = 0$$

$$16c''(t)e^{-3t/4} \cancel{-12c'(t)e^{-3t/4}} \cancel{-12c'(t)e^{-3t/4}} \cancel{+9c(t)e^{-3t/4}} \cancel{+24c'(t)e^{-3t/4}} \cancel{-18c(t)e^{-3t/4}} \cancel{+9c(t)e^{-3t/4}} = 0$$

$$16c''(t)e^{-3t/4} = 0$$

Divide both sides by $16e^{-3t/4}$.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

Therefore, the general solution is

$$y(t) = C_1 te^{-3t/4} + C_2 e^{-3t/4}.$$