Problem 8

In each of Problems 1 through 10, find the general solution of the given differential equation.

\[ 16y'' + 24y' + 9y = 0 \]

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form \( y = e^{rt} \).

\[ y = e^{rt} \rightarrow y' = re^{rt} \rightarrow y'' = r^2e^{rt} \]

Substitute these expressions into the ODE.

\[ 16(r^2e^{rt}) + 24(re^{rt}) + 9(e^{rt}) = 0 \]

Divide both sides by \( e^{rt} \).

\[ 16r^2 + 24r + 9 = 0 \]

\[ (4r + 3)^2 = 0 \]

\[ r = \left\{ \begin{array}{l} -\frac{3}{4} \end{array} \right\} \]

One solution to the ODE is \( y = e^{-3t/4} \). Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, \( y = ce^{-3t/4} \). According to the method of reduction of order, the general solution is found by allowing \( c \) to vary as a function of \( t \).

\[ y(t) = c(t)e^{-3t/4} \]

Substitute this expression for \( y \) into the original ODE to determine \( c(t) \).

\[ 16y'' + 24y' + 9y = 0 \rightarrow 16[c(t)e^{-3t/4}]'' + 24[c(t)e^{-3t/4}]' + 9[c(t)e^{-3t/4}] = 0 \]

Evaluate the derivatives using the product rule.

\[
16 \left[ c'(t)e^{-3t/4} - \frac{3}{4}c(t)e^{-3t/4} \right]' + 24 \left[ c'(t)e^{-3t/4} - \frac{3}{4}c(t)e^{-3t/4} \right]' + 9 \left[ c(t)e^{-3t/4} \right]' = 0 \\
16 \left[ c''(t)e^{-3t/4} - \frac{3}{4}c'(t)e^{-3t/4} - \frac{3}{4}c(t)e^{-3t/4} + \frac{9}{16}c(t)e^{-3t/4} \right] + 24 \left[ c'(t)e^{-3t/4} - \frac{3}{4}c(t)e^{-3t/4} \right]' + 9[c(t)e^{-3t/4}] = 0 \\
16c''(t)e^{-3t/4} - 12c'(t)e^{-3t/4} - 12c(t)e^{-3t/4} + 9c(t)e^{-3t/4} + 24c'(t)e^{-3t/4} - 18c(t)e^{-3t/4} + 9c(t)e^{-3t/4} = 0 \\
16c''(t)e^{-3t/4} = 0 \\
\]

Divide both sides by \( 16e^{-3t/4} \).

\[ c''(t) = 0 \]

Integrate both sides with respect to \( t \).

\[ c'(t) = C_1 \]

Integrate both sides with respect to \( t \) once more.

\[ c(t) = C_1t + C_2 \]

Therefore, the general solution is

\[ y(t) = C_1te^{-3t/4} + C_2e^{-3t/4}. \]

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