Problem 9

In each of Problems 1 through 10, find the general solution of the given differential equation.

$$25y'' - 20y' + 4y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$25(r^2e^{rt}) - 20(re^{rt}) + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$25r^2 - 20r + 4 = 0$$

$$(5r-2)^2 = 0$$

$$r = \left\{\frac{2}{5}\right\}$$

One solution to the ODE is $y = e^{2t/5}$. Because the ODE is homogeneous, any constant multiple of this is also a solution, that is, $y = ce^{2t/5}$. According to the method of reduction of order, the general solution is found by allowing c to vary as a function of t.

$$y(t) = c(t)e^{2t/5}$$

Substitute this expression for y into the original ODE to determine $c(t)$.

$$25y'' - 20y' + 4y = 0 \quad \rightarrow \quad 25[c(t)e^{2t/5}]'' - 20[c(t)e^{2t/5}]' + 4[c(t)e^{2t/5}] = 0$$

Evaluate the derivatives using the product rule.

$$25\left[c'(t)e^{2t/5} + \frac{2}{5}c(t)e^{2t/5}\right]' - 20\left[c'(t)e^{2t/5} + \frac{2}{5}c(t)e^{2t/5}\right] + 4[c(t)e^{2t/5}] = 0$$

$$25\left[c''(t)e^{2t/5} + \frac{2}{5}c'(t)e^{2t/5} + \frac{2}{5}c'(t)e^{2t/5} + \frac{4}{25}c(t)e^{2t/5}\right] - 20\left[c'(t)e^{2t/5} + \frac{2}{5}c(t)e^{2t/5}\right] + 4[c(t)e^{2t/5}] = 0$$

$$25c''(t)e^{2t/5} + \cancel{10c'(t)e^{2t/5}} + \cancel{10c'(t)e^{2t/5}} + \cancel{4c(t)e^{2t/5}} - \cancel{20c'(t)e^{2t/5}} - \cancel{8c(t)e^{2t/5}} + \cancel{4c(t)e^{2t/5}} = 0$$

$$25c''(t)e^{2t/5} = 0$$

Divide both sides by $25e^{2t/5}$.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1t + C_2$$

Therefore, the general solution is

$$y(t) = C_1te^{2t/5} + C_2e^{2t/5}.$$