Problem 10

In each of Problems 1 through 10, find the general solution of the given differential equation.

$$2y'' + 2y' + y = 0$$

Solution

Since this is a linear homogeneous constant-coefficient ODE, the solution is of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$2(r^2 e^{rt}) + 2(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$2r^2 + 2r + 1 = 0$$

$$r = \frac{-2 \pm \sqrt{4 - 4(2)(1)}}{2(2)} = \frac{-2 \pm \sqrt{-4}}{4} = \frac{-2 \pm 2i}{4} = -\frac{1}{2} \pm \frac{i}{2}$$

$$r = \left\{-\frac{1}{2} - \frac{i}{2}, -\frac{1}{2} + \frac{i}{2}\right\}$$

Two solutions to the ODE are $y = e^{(-1/2-i/2)t}$ and $y = e^{(-1/2+i/2)t}$, so the general solution is a linear combination of the two.

$$\begin{aligned}
y(t) &= C_1 e^{(-1/2-i/2)t} + C_2 e^{(-1/2+i/2)t} \\
&= C_1 e^{-t/2-it/2} + C_2 e^{-t/2+it/2} \\
&= C_1 e^{-t/2} e^{-it/2} + C_2 e^{-t/2} e^{it/2} \\
&= C_1 e^{-t/2}\left[\cos\left(-\frac{t}{2}\right) + i\sin\left(-\frac{t}{2}\right)\right] + C_2 e^{-t/2}\left[\cos\left(\frac{t}{2}\right) + i\sin\left(\frac{t}{2}\right)\right] \\
&= C_1 e^{-t/2}\left[\cos\left(\frac{t}{2}\right) - i\sin\left(\frac{t}{2}\right)\right] + C_2 e^{-t/2}\left[\cos\left(\frac{t}{2}\right) + i\sin\left(\frac{t}{2}\right)\right] \\
&= C_1 e^{-t/2}\cos\left(\frac{t}{2}\right) - iC_1 e^{-t/2}\sin\left(\frac{t}{2}\right) + C_2 e^{-t/2}\cos\left(\frac{t}{2}\right) + iC_2 e^{-t/2}\sin\left(\frac{t}{2}\right) \\
&= (C_1 + C_2)e^{-t/2}\cos\left(\frac{t}{2}\right) + (-iC_1 + iC_2)e^{-t/2}\sin\left(\frac{t}{2}\right)
\end{aligned}$$

Therefore, using C_3 for $C_1 + C_2$ and C_4 for $-iC_1 + iC_2$,

$$y(t) = C_3 e^{-t/2}\cos\left(\frac{t}{2}\right) + C_4 e^{-t/2}\sin\left(\frac{t}{2}\right).$$